

Desingularization of Coassociative 4-folds with Conical Singularities II: Obstructions and Applications

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Abstract

We study the problem of desingularizing coassociative conical singularities via gluing, allowing for topological and analytic obstructions, and discuss applications. This extends work in [17] on the unobstructed case. We interpret the analytic obstructions geometrically via the obstruction theory for deformations of conically singular coassociative 4-folds, and thus relate them to the stability of the singularities. We use our results to describe the relationship between moduli spaces of coassociative 4-folds with conical singularities and those of their desingularizations. We also apply our theory in examples, including to the known conically singular coassociative 4-folds in compact holonomy G_2 manifolds.

1 Introduction

Coassociative 4-folds are calibrated 4-dimensional submanifolds in 7-manifolds with exceptional holonomy G_2 (and, more generally, in 7-manifolds with a G_2 structure). Studying gluing problems for calibrated submanifolds in manifolds with special holonomy has proven to be a rich and fruitful avenue of research, particularly in special Lagrangian geometry in the work of Joyce [9, 10], Haskins and Kapouleas [5] and Pacini [26], as well as for associative [24] and coassociative geometry [17]. In particular, the desingularization problem for calibrated submanifolds with conical singularities naturally feeds into the understanding of the boundary of the moduli space of smooth calibrated submanifolds, and has crucial consequences for the construction of potential invariants for manifolds

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with special holonomy by suitable “counting” of calibrated submanifolds (see, for example, [7, 8] for a discussion of these issues in the special Lagrangian case).

In any gluing problem one naturally has to tackle the issue of *obstructions*. This is normally achieved by making strong assumptions on the geometry of the submanifolds to be glued, for example in [9, 17, 26], or by restricting to situations where the obstructions can be identified and resolved in a natural way, either via topological conditions (as in [10]) or symmetries of the problem (as in [5]). In this article we extend the work in [17] and consider a gluing problem in coassociative geometry where we deal with both topological and analytic obstructions in desingularizing isolated conical singularities. We interpret the analytic obstructions geometrically using the deformation theories of asymptotically conical and conically singular coassociative 4-folds developed in [16, 18]. We thus relate our obstructions to the notion of *stability* of coassociative conical singularities introduced in [20].

We use our desingularization results to help describe how the moduli spaces of asymptotically conical, conically singular and smooth compact coassociative 4-folds are related, and thus provide a greater understanding of the boundary of the moduli space of compact coassociative 4-folds. In the case of stable conical singularities, this enables us to construct a local diffeomorphism between the gluing data associated to the singular coassociative 4-fold and a neighbourhood of the coassociative smoothing “near the boundary” of the moduli space.

We also discuss examples where our desingularization theory applies, including the first known examples of coassociative 4-folds with conical singularities in compact manifolds with G_2 holonomy, which were constructed in [20].

The setting in this article is the following. We have a coassociative 4-fold N in an almost G_2 manifold M (a 7-manifold with a closed G_2 structure) and we suppose that N has a single conical singularity z modelled on a cone C . We also assume there exists a coassociative 4-fold A in \mathbb{R}^7 which is asymptotically conical with rate $\lambda < -\frac{1}{2}$ to the cone C . (See §2 for precise definitions.)

In Definition 3.4 we give a *matching condition* between A and N . The matching condition is a mixture of topological and analytic constraints, which then allows us to deal with both topological and analytic obstructions.

To give a sense of the matching condition we make some observations. Let Σ be the link of C and let $j_2^A : H^2(A) \rightarrow H^2(\Sigma)$ and $j_2^N : H^2(\hat{N}) \rightarrow H^2(\Sigma)$ be the induced maps arising from inclusion of Σ in A and $\hat{N} = N \setminus \{z\}$ (the non-compact manifold given by removing the singularity from N).

If φ_0 is the standard G_2 structure on \mathbb{R}^7 , then since φ_0 is closed and $\varphi_0|_A = 0$

we have an element $[\varphi_0] \in H^3(\mathbb{R}^7, A) \cong H^2(A)$, which we may also view as the cohomology class of the infinitesimal deformation of A corresponding to dilation. The natural topological constraint is therefore that $j_2^A[\varphi_0]$ lies in $\text{Im } j_2^N$.

We can relate the analytic obstructions to the obstruction theory of N and thus to the notion of \mathcal{C} -stability of the cone C (see Definition 3.19) for a deformation family \mathcal{C} of C – this condition is discussed in detail in [20].

Overall, we have the following interpretation of the matching condition.

Proposition 1.1 *If $j_2^A[\varphi_0] \in \text{Im } j_2^N$ and the cone C at the singularity is \mathcal{C} -stable, then the matching condition is satisfied.*

We show that the matching condition allows us to desingularize N using A via gluing, giving our main result.

Theorem 1.2 *If the matching condition is satisfied, there exists $\tau > 0$ such that for all $t \in (0, \tau)$ there is a smooth compact coassociative 4-fold $N(t)$ in M , formed by gluing tA and N , such that $N(t)$ converges to N in the sense of currents as $t \rightarrow 0$.*

By deforming $N(t)$ we obtain a family of desingularizations of N whose dimension we can determine from the topology of A and N . Recall that for a noncompact 4-manifold we define b_+^2 by considering the cup product on cohomology classes representing by compactly supported 2-forms.

Corollary 1.3 *If the matching condition is satisfied, we have a smooth family of nearby compact coassociative smoothings of N of dimension $b_+^2(A) + b_+^2(\hat{N}) + \dim(\text{Im } j_2^A \cap \text{Im } j_2^N)$.*

This shows that the moduli space of smooth compact coassociative 4-folds can be non-compact and that coassociative 4-folds with conical singularities can arise on the boundary of the moduli space. Moreover, we have a *gluing map* from the moduli space of matching pairs (N, tA) into the moduli space of smooth compact coassociative 4-folds, from which we may deduce the following.

Proposition 1.4 *If C is stable, the gluing map is a local diffeomorphism.*

The organisation of the paper is as follows.

- In §2 we provide the basic definitions and notation which shall be used throughout the paper and discuss foundational results for coassociative 4-folds involving self-dual 2-forms and tubular neighbourhood theorems.

- In §3 we describe our gluing construction, identify the obstructions to the procedure and the necessary matching condition for the construction to succeed. We also set up the analytic framework for our problem and discuss the relationship between our matching conditions and the deformation theory of A and N , which allows us to prove Proposition 1.1.
- In §4, we realise our smoothing of the singular coassociative 4-fold as the fixed point of a map between Banach spaces, which we prove is a contraction by deriving appropriate analytic estimates on the smoothing using estimates on the “building blocks” A and N . We deduce Theorem 1.2 and Corollary 1.3 from this work.
- In §5 we compare the moduli space of “matching pairs” (N, tA) to the moduli space of smoothings and deduce Proposition 1.4. We conclude by applying our theory in examples.

2 Foundations

In this section we describe the basic foundational material we need to tackle our desingularization problem.

2.1 Basic definitions

There exists a 3-form φ_0 on \mathbb{R}^7 with constant coefficients such that $\text{Stab}(\varphi_0) \subseteq \text{GL}(7, \mathbb{R})$ is isomorphic to G_2 . In fact $\text{Stab}(\varphi_0) \subseteq \text{SO}(7)$ so G_2 preserves the Euclidean metric g_0 on \mathbb{R}^7 .

Definition 2.1 We call a 3-form φ on an oriented 7-manifold M a G_2 structure if, for all $x \in M$, $\varphi|_x = \iota_x^*(\varphi_0)$ for some orientation preserving isomorphism $\iota_x : T_x M \rightarrow \mathbb{R}^7$. A G_2 structure φ defines a metric g_φ on M .

We denote a 7-manifold M endowed with a G_2 structure φ by (M, φ) . An oriented 7-manifold will admit a G_2 structure if and only if it is spin. We now define special classes of G_2 structures which will be especially relevant.

Definition 2.2 We say that (M, φ) is an *almost G_2 manifold* if $d\varphi = 0$. We call (M, φ) a G_2 manifold if $d\varphi = d^*\varphi = 0$ with respect to g_φ , which is equivalent to saying that the holonomy $\text{Hol}(g_\varphi) \subseteq G_2$.

In 7-manifolds with a G_2 structure we have a distinguished class of submanifolds which shall form the basis for our study.

Definition 2.3 A *coassociative 4-fold* X in (M, φ) is a 4-dimensional submanifold of M such that $\varphi|_X \equiv 0$, oriented so that $*\varphi|_X > 0$. Equivalently, X is coassociative if and only if $*\varphi|_X = \text{vol}_X$.

When (M, φ) is a G_2 manifold, coassociative 4-folds are volume-minimizing in their homology class. Although coassociative 4-folds lose this property in general almost G_2 manifolds, their geometry otherwise has essentially the same features as in the G_2 manifold case. Coassociative geometry is discussed in detail in [11].

Let $B(0; r)$ denote the Euclidean ball of radius r about 0 in \mathbb{R}^7 . For a cone C in \mathbb{R}^7 (i.e. a dilation-invariant subset) such that $C \setminus \{0\}$ is a smooth submanifold we let $\Sigma = C \cap \mathcal{S}^6$ with the induced metric g_Σ , let $\iota : C \cong \mathbb{R}^+ \times \Sigma \rightarrow \mathbb{R}^7$ be the inclusion map and let ∇_C be the Levi-Civita connection of the cone metric $g_C = dr^2 + r^2 g_\Sigma$ on C .

We may now define the two types of submanifold which shall appear in our desingularization problem: namely *conically singular* and *asymptotically conical*. In each case we have a noncompact submanifold of (M, φ) which, outside a compact set, is diffeomorphic to a cone (or finite collection of cones) and converges to the cone with some prescribed rate. The two classes of submanifold will be dual in the sense that one converges towards the cone near its vertex (so has a singular point) whereas the other converges to the cone near infinity.

Definition 2.4 Let N be a (singular) submanifold of (M, φ) and let $z \in N$. Choose local coordinates $\chi : B(0; \epsilon_M) \rightarrow V \ni z$, for some $\epsilon_M \in (0, 1)$ and open $V \subseteq M$, such that $\chi(0) = z$ and $(d\chi|_0)^*(\varphi|_z, g_\varphi|_z) = (\varphi_0, g_0)$. (These are natural coordinates for a neighbourhood of z in (M, φ) .)

We say that N has a *conical singularity* at z if there exist a cone $C \subseteq \mathbb{R}^7$ with link $\Sigma \subseteq \mathcal{S}^6$, constants $\epsilon \in (0, \epsilon_M)$ and $\mu \in (1, 2)$, open $U \subseteq V \cap N$ containing z and a smooth map

$$\Phi_N : (0, \epsilon) \times \Sigma \rightarrow B(0; \epsilon_M) \quad \text{such that} \quad \Psi_N = \chi \circ \Phi_N : (0, \epsilon) \times \Sigma \rightarrow U \setminus \{z\}$$

is a diffeomorphism and

$$|\nabla_C^j (\Phi_N(r, \sigma) - \iota(r, \sigma))| = O(r^{\mu-j}) \quad \text{for } j \in \mathbb{N} \text{ as } r \rightarrow 0 \text{ on } C. \quad (1)$$

We call C the *cone* and μ the *rate* at the singularity.

We say that N is a *conically singular* (CS) submanifold if N is compact and connected and smooth except for finitely many conical singularities. We call N a *CS coassociative 4-fold* if N is a CS submanifold whose nonsingular part is a coassociative 4-fold.

Remarks

- (a) By [16, Proposition 3.6], if N is a CS coassociative 4-fold then the cones at the singularities are coassociative in \mathbb{R}^7 .
- (b) The stipulation that $\mu < 2$ allows the definition of conical singularity to be essentially independent of the choice of local coordinates χ , as explained in [16, §3.2].
- (c) Notice that if N is CS with rate μ_0 it is also CS with any rate $\mu \in (1, \mu_0]$. We are thus free to reduce the rate μ , so we always choose μ close to 1.

Definition 2.5 A (smooth) submanifold A of \mathbb{R}^7 is an *asymptotically conical* (AC) submanifold if there exist a cone $C \subseteq \mathbb{R}^7$ with link $\Sigma \subseteq \mathcal{S}^6$, constants $R > 0$ and $\lambda < 1$, compact $K_A \subseteq A$ and a diffeomorphism $\Phi_A : (R, \infty) \times \Sigma \rightarrow A \setminus K_A$ satisfying

$$|\nabla_C^j (\Phi_A(r, \sigma) - \iota(r, \sigma))| = O(r^{\lambda-j}) \quad \text{for } j \in \mathbb{N} \text{ as } r \rightarrow \infty \text{ on } C. \quad (2)$$

We say that A is AC with *rate* λ to C to emphasise the choice of C and λ .

Remarks

- (a) By [18, Proposition 2.8], if $A \subseteq \mathbb{R}^7$ is coassociative and AC to C then C is coassociative.
- (b) Observe that A only genuinely converges to C at infinity if $\lambda < 0$, so allowing for $\lambda \in [0, 1)$ permits weak decay.
- (c) Note that if A is AC with rate λ_0 it is also AC with any higher rate $\lambda \in [\lambda_0, 1)$, so we are at liberty to increase the rate λ .

As we see, AC submanifolds are smoothings of cones and thus provide obvious models for desingularizing CS submanifolds via gluing. However, the challenge is to desingularize CS *coassociative* 4-folds so that the coassociative condition is preserved, so one would naturally require AC coassociative 4-folds in the gluing. Our problem is to study when this approach may be successfully applied and understand the obstructions to the coassociative gluing process.

To fix notation we describe the ingredients we wish to feed into our problem.

- Let C be a coassociative cone in $(\mathbb{R}^7, \varphi_0)$ with link $\Sigma = C \cap \mathcal{S}^6$.

- Let N be a CS coassociative 4-fold in an almost G_2 manifold (M, φ) with a single conical singularity at z with rate μ and cone C .

Choose local coordinates $\chi : B(0; \epsilon_M) \rightarrow M$ about z as in Definition 2.4. By Definition 2.4, there exist $\epsilon \in (0, \epsilon_M)$, compact $K_N \subseteq \hat{N} = N \setminus \{z\}$ and a smooth map

$$\Phi_N : (0, \epsilon) \times \Sigma \rightarrow B(0; \epsilon_M) \quad \text{such that} \quad \Psi_N = \chi \circ \Phi_N : (0, \epsilon) \times \Sigma \rightarrow \hat{N} \setminus K_N$$

is a diffeomorphism satisfying (1). We can choose Φ_N such that

$$\Phi_N(r, \sigma) - \iota(r, \sigma) \in (T_{r\sigma}C)^\perp.$$

- Let $A \subseteq \mathbb{R}^7$ be a coassociative 4-fold which is AC with rate $\lambda < 1$ to C .

By Definition 2.5, there exist $R > 0$, compact $K_A \subseteq A$, and a diffeomorphism $\Phi_A : (R, \infty) \times \Sigma \rightarrow A \setminus K_A$ satisfying (2). We can also choose Φ_A so that

$$\Phi_A(r, \sigma) - \iota(r, \sigma) \in (T_{r\sigma}C)^\perp.$$

Whenever t is sufficiently small that $t^{-1}\epsilon > R$, we set

$$\hat{A}(t) = K_A \cup \Phi_A((R, t^{-1}\epsilon) \times \Sigma).$$

The subsets $t\hat{A}(t) \subseteq tA$ will be glued to \hat{N} to resolve the singularity z .

- Let $\tau \in (0, 1)$ be small enough that $\tau R < \epsilon$ and $\tau\hat{A}(\tau) \subseteq B(0; \epsilon_M)$. Throughout we let $t \in (0, \tau)$ be arbitrary and will make τ smaller a finite number of times, continuing to refer to this new smaller constant as τ . Our constraints on τ ensure that we can use the local coordinates χ to view the gluing of $t\hat{A}(t)$ to \hat{N} , outside the compact set K_N , as occurring in $B(0; \epsilon_M) \subseteq \mathbb{R}^7$.

We shall occasionally refer to $\hat{N} \setminus K_N$ and $A \setminus K_A$ as the end (or ends since they could be disconnected) of \hat{N} and A .

2.2 Self-dual 2-forms and tubular neighbourhoods

In geometric gluing problems, it is often crucial to know the relationship between deformations of the building blocks and those of the glued object. We shall therefore need to understand deformations of coassociative 4-folds, for which the key result is the following [23, c.f. Proposition 4.2].

Proposition 2.6 *Let X be a coassociative 4-fold in (M, φ) . There is an isometric isomorphism j_X between the normal bundle $\nu(X)$ of X in M and $\Lambda_+^2 T^*X$ given by $v \mapsto (v \lrcorner \varphi)|_X$.*

Note For any coassociative 4-fold X we will consistently use the notation j_X to indicate the isomorphism in Proposition 2.6.

Using this identification, we can view nearby submanifolds to X as graphs of small self-dual 2-forms; that is, give open neighbourhoods of the zero section in $\Lambda_+^2 T^*X$ and of X in M and a diffeomorphism between them which acts as the identity on X (identified with the zero section as usual). However, we must perform this construction carefully so as to ensure compatibility with j_X and to take into account the asymptotic behaviour of A and N . We first make the compatibility property precise.

Definition 2.7 Let X be a coassociative 4-fold in (M, φ) . Suppose we have a smooth map Υ_X from an open neighbourhood of the zero section in $\Lambda_+^2 T^*X$ to an open tubular neighbourhood of X in M , acting as the identity id_X on X . We may then view $d\Upsilon_X|_X$ as a map from $TX \oplus \Lambda_+^2 T^*X$ to $TX \oplus \nu(X)$. We say that Υ_X is *compatible with j_X* if

$$d\Upsilon_X|_X = \begin{pmatrix} I & \mathcal{A} \\ 0 & j_X^{-1} \end{pmatrix},$$

where $I : TX \rightarrow TX$ is the identity and $\mathcal{A} : \Lambda_+^2 T^*X \rightarrow TX$ is arbitrary.

We now construct our tubular neighbourhoods using self-dual 2-forms as in the author's earlier papers [16, 17, 18], however our presentation is different and more in the style of [10, §3-4] as it is more convenient.

Observe that if $\alpha \in C^\infty(\Lambda_+^2 T^*C)$ then $|\alpha|_{t^2 g_C} = t^{-2} |\alpha|_{g_C}$ for $t > 0$, so $|t^3 \alpha|_{t^2 g_C} = t |\alpha|_{g_C}$. Therefore the natural dilation action on $\Lambda_+^2 T^*C$ is given by:

$$(r, \sigma, \alpha(r, \sigma)) \mapsto (tr, \sigma, t^3 \alpha(tr, \sigma)). \quad (3)$$

Proposition 2.8

- (a) *There exist dilation-invariant open neighbourhoods $U_C \subseteq \Lambda_+^2 T^*C$ and $T_C \subseteq \mathbb{R}^7$ of C , with U_C given by*

$$U_C = \{(r, \sigma, \alpha(r, \sigma)) : |\alpha| < 2\zeta\}$$

for some $\zeta > 0$, and a dilation-equivariant diffeomorphism $\Upsilon_C : U_C \rightarrow T_C$ such that $\Upsilon_C|_C = \text{id}_C$ and is compatible with j_C .

(b) Make ϵ smaller and R larger if necessary so that

$$|\Phi_N(r, \sigma) - r\sigma| < \zeta \text{ for all } r < \epsilon \quad \text{and} \quad |\Phi_A(r, \sigma) - r\sigma| < \zeta \text{ for all } r > R.$$

There exist self-dual 2-forms α_N on $(0, \epsilon) \times \Sigma$ and α_A on $(R, \infty) \times \Sigma$ such that

$$\Upsilon_C(r, \sigma, \alpha_N(r, \sigma)) = \Phi_N(r, \sigma) - r\sigma \text{ and } \Upsilon_C(r, \sigma, \alpha_A(r, \sigma)) = \Phi_A(r, \sigma) - r\sigma.$$

Moreover, for all $j \in \mathbb{N}$,

$$|\nabla_C^j \alpha_N| = O(r^{\mu-j}) \text{ as } r \rightarrow 0 \quad \text{and} \quad |\nabla_C^j \alpha_A| = O(r^{\lambda-j}) \text{ as } r \rightarrow \infty.$$

Proof: Applying the Tubular Neighbourhood Theorem to $\Sigma \subseteq \mathcal{S}^6$, we can easily construct diffeomorphic dilation-invariant open neighbourhoods of C in $\nu(C)$ and \mathbb{R}^7 . Using j_C gives (a).

We can certainly change ϵ and R as claimed given the asymptotic behaviour of Φ_N and Φ_A . Part (b) then follows from (a), the definition of N and A as CS and AC submanifolds and the fact that j_C is an isometric isomorphism. \square

Proposition 2.8 says we may effectively view the ends of A and N as graphs of the self-dual 2-forms α_A and α_N on the cone. We can then extend this result as in our earlier work to give neighbourhoods of A and N , which are adapted so that we may realize graphs of small self-dual 2-forms on the ends as graphs of small self-dual 2-forms on the cone.

Proposition 2.9 *Recall the notation of Proposition 2.8.*

- (a) *There exist open neighbourhoods $U_N \subseteq \Lambda_+^2 T^* \hat{N}$ and $T_N \subseteq M$ of \hat{N} and a diffeomorphism $\Upsilon_N : U_N \rightarrow T_N$ such that $\Upsilon_N|_N = \text{id}_N$ and is compatible with j_N . Further,*

$$\Psi_N^*(U_N) = \{(r, \sigma, \alpha(r, \sigma)) : r < \epsilon, |\alpha| < \zeta\}$$

and

$$\Upsilon_N(\Psi_N(r, \sigma), \alpha(\Psi_N(r, \sigma))) = \chi \circ \Upsilon_C(r, \sigma, \alpha_N(r, \sigma) + \Psi_N^* \alpha(r, \sigma)).$$

- (b) *There exist open neighbourhoods $U_A \subseteq \Lambda_+^2 T^* A$ and $T_A \subseteq \mathbb{R}^7$ of A and a diffeomorphism $\Upsilon_A : U_A \rightarrow T_A$ such that $\Upsilon_A|_A = \text{id}_A$ and is compatible with j_A . Further,*

$$\Phi_A^*(U_A) = \{(r, \sigma, \alpha(r, \sigma)) : r > R, |\alpha| < \zeta\}$$

and

$$\Upsilon_A(\Phi_A(r, \sigma), \alpha(\Phi_A(r, \sigma))) = \Upsilon_C(r, \sigma, \alpha_A(r, \sigma) + \Phi_A^* \alpha(r, \sigma)).$$

Remark Observe the important difference between (a) and (b): in (a) we need to use the particular identification χ between an open ball in \mathbb{R}^7 and an open neighbourhood of z in M .

Having identified self-dual 2-forms α with nearby submanifolds X_α to a coassociative 4-fold X we may ask: what is the condition on α which makes X_α coassociative? By Definition 2.3 this is given by $\varphi|_{X_\alpha} = 0$, which leads to a fully nonlinear equation on α . By the calculation in [23, p. 731] we see that the linearisation of this equation is $d\alpha = 0$ since $d\varphi = 0$. (Here is where we use the condition that (M, φ) is an almost G_2 manifold, since otherwise the linearisation would have further terms.) We deduce the following well-known fact.

Proposition 2.10 *Let X be a coassociative 4-fold in an almost G_2 manifold. Infinitesimal coassociative deformations of X are given by closed self-dual 2-forms on X .*

Closed self-dual forms are trivially also coclosed. Hence, if X is compact, Hodge theory implies that such forms uniquely represent cohomology classes in $H^2(X)$. In the non-compact setting we do not have such a result, but for AC and CS 4-folds we can say which cohomology classes are uniquely represented by L^2 closed self-dual 2-forms. This leads to our next definition.

Definition 2.11 For any Riemannian 4-manifold X , let

$$\begin{aligned}\mathcal{H}^2(X) &= \{\alpha \in L^2(\Lambda^2 T^*X) : d\alpha = d^*\alpha = 0\}, \\ \mathcal{H}_\pm^2(X) &= \{\alpha \in L^2(\Lambda_\pm^2 T^*X) : d\alpha = 0\}.\end{aligned}$$

Notice that $\mathcal{H}^2(X) = \mathcal{H}_+^2(X) \oplus \mathcal{H}_-^2(X)$ and that by elliptic regularity $\mathcal{H}^2(X)$ consists of smooth forms.

If X is compact then $\dim \mathcal{H}^2(X) = b^2(X)$ and $\dim \mathcal{H}_\pm^2(X) = b_\pm^2(X)$. If X is an AC or (the nonsingular part of) a CS 4-fold and we let

$$\mathcal{J}(X) = \text{Im} (H_{\text{cs}}^2(X) \rightarrow H^2(X))$$

then, by [14, Examples (0.15) & (0.16)],

$$\dim \mathcal{H}^2(X) = \dim \mathcal{J}(X) \quad \text{and} \quad \dim \mathcal{H}_\pm^2(X) = \dim \mathcal{J}_\pm(X),$$

where $\mathcal{J}_\pm(X)$ are the maximal positive and negative subspaces of $\mathcal{J}(X)$ with respect to the cup product. (The subspaces $\mathcal{J}_\pm(X)$ are well-defined because the cohomology classes in $\mathcal{J}(X)$ are represented by compactly supported forms.) We thus define $b_\pm^2(X) = \dim \mathcal{J}_\pm(X)$.

By [23, §4], the deformation theory of compact coassociative 4-folds X is unobstructed, so infinitesimal deformations always extend to genuine deformations and thus we have the following.

Theorem 2.12 *Let X be a compact coassociative 4-fold in an almost G_2 manifold. The moduli space of compact coassociative deformations of X is a smooth manifold near X of dimension $b_+^2(X)$.*

The author extended this result in [16] and [18] to the CS and AC settings, where various similarities and differences occur which shall be discussed later. These results will be crucial in understanding obstructions to the gluing problem.

3 Desingularization: geometry

In this section we tackle the more “geometric” aspects of our desingularization problem. The key part is to construct an appropriate connect sum $\tilde{N}(t)$ of \hat{N} and tA such that $\tilde{N}(t)$ is a smooth compact 4-fold with $\tilde{N}(t) \rightarrow N$ as $t \rightarrow 0$. The crucial point will be to ensure that $\tilde{N}(t)$ is sufficiently “close” to being coassociative; i.e. that $|\varphi|_{\tilde{N}(t)}$ is “small enough” that one may hope to perturb $\tilde{N}(t)$ to a nearby coassociative 4-fold $N(t)$.

Unlike in [17], where one was able to construct $\tilde{N}(t)$ using a rather naive connect sum, here we have to use a more refined technique which requires a detailed understanding of the geometric *obstructions* to the coassociative gluing procedure. We discover that the obstructions which emerge are both topological and analytic in nature, and we can give natural interpretations for the obstructions.

3.1 Obstructions

Studying the argument in [17], one sees that for our problem we simply cannot use the same method of constructing $\tilde{N}(t)$ since the analysis will fail. This is not a flaw with the analytic method, but rather it is a *geometric* phenomenon. Specifically, in [17] geometric assumptions were made precisely to ensure that the desingularization was *unobstructed*. In general, there are geometric *obstructions* to resolving the coassociative conical singularity, which we now identify.

We begin by examining the cone C . Consider a self-dual 2-form α on C which is homogeneous of rate v say. We may write

$$\alpha = r^{v+2}(\alpha_\Sigma + r^{-1}dr \wedge *_\Sigma \alpha_\Sigma)$$

for a 2-form α_Σ on the link Σ of C , noting that $|\alpha_\Sigma|_{g_C} = O(r^{-2})$. (We use the notation $*_\Sigma$ to clarify that we are using the Hodge star on Σ .) The condition that α is closed is equivalent to

$$d*_\Sigma \alpha_\Sigma = (v+2)\alpha_\Sigma \quad \text{and} \quad d\alpha_\Sigma = 0. \quad (4)$$

Such closed forms α define infinitesimal coassociative deformations of C by Proposition 2.10. These forms will also naturally relate to deformations of A and \hat{N} . To understand this relationship we first make a convenient definition.

Definition 3.1 For $v \in \mathbb{R}$ let $D(v) \subseteq C^\infty(\Lambda^2 T^* \Sigma)$ be the space of solutions to (4), so $D(v)$ corresponds to the homogeneous closed self-dual 2-forms on C of rate v . We also let $\mathcal{D} = \{v \in \mathbb{R} : D(v) \neq \{0\}\}$ and let $d_{\mathcal{D}}(v) = \dim D(v)$.

Remarks

- (a) The set \mathcal{D} is countable and discrete, and $d_{\mathcal{D}}(v)$ is always finite.
- (b) For $v = -2$, (4) is equivalent to the statement that α_Σ is closed and coclosed. Thus $d_{\mathcal{D}}(-2) = b^1(\Sigma)$ and $-2 \in \mathcal{D}$ if and only if $b^1(\Sigma) \neq 0$.

Since we may view A as a manifold with boundary Σ , we have the following exact sequence:

$$\cdots \longrightarrow H_{\text{cs}}^m(A) \xrightarrow{\iota_m^A} H^m(A) \xrightarrow{j_m^A} H^m(\Sigma) \xrightarrow{\partial_m^A} H_{\text{cs}}^{m+1}(A) \longrightarrow \cdots \quad (5)$$

The connection between deformations of C and A can now be succinctly expressed through one of the main results in [18].

Theorem 3.2 *Suppose that the rate $\lambda < 0$ and let $\lambda_+ \in (-2, 0) \setminus \mathcal{D}$ be such that $\lambda_+ \geq \lambda$. The moduli space of deformations of A as an AC coassociative 4-fold with rate λ_+ and cone C is a smooth manifold near A of dimension*

$$b_+^2(A) + \dim \text{Im } j_2^A + \sum_{v \in (-2, \lambda_+)} d_{\mathcal{D}}(v),$$

which is the dimension of

$$\{\alpha \in C^\infty(\Lambda_+^2 T^* A) : d\alpha = 0, |\nabla_C^j \Phi_A^* \alpha| = O(r^{\lambda_+ - j}) \text{ as } r \rightarrow \infty \text{ for all } j \in \mathbb{N}\}.$$

The appearance of the term $b_+^2(A)$ is clear by Definition 2.11. A key part of the proof and dimension count relies on showing that various closed self-dual 2-forms on C lift to A , applying the theory in [15]. The forms on C corresponding

to forms in $\mathcal{H}_+^2(A)$ are actually zero, but for the other terms in the dimension count one has non-trivial forms on C lifting to A .

Specifically, the harmonic representatives of the classes in $\text{Im } j_2^A$ define the homogeneous closed self-dual 2-forms on C with order $O(r^{-2})$ which lift to define closed self-dual 2-forms on A . Notice that such forms on A , given their decay rate on the ends, cannot lie in L^2 and so do not contribute to $b_+^2(A)$.

Moreover, the sum over $d_{\mathcal{D}}(v)$ counts the homogeneous closed self-dual 2-forms on C which have rate between -2 and λ_+ , and the proof of Theorem 3.2 shows that these forms on C all lift to closed self-dual 2-forms on A .

The final key part of the proof of Theorem 3.2 is to show that, given a closed self-dual 2-form α^0 on A with appropriate decay on the ends, one can solve for a transverse self-dual 2-form α' on A so that φ_0 vanishes on the graph of $\alpha^0 + \alpha'$ via the Implicit Function Theorem. Thus we can extend the infinitesimal AC coassociative deformation of A given by α^0 to a genuine deformation.

With these preliminaries we are now able to analyse A further.

Proposition 3.3 *Suppose that $\lambda < 0$ and let*

$$\mathcal{K}_C(\lambda) = \begin{cases} \{r^{v+2}(\alpha_\Sigma + r^{-1}dr \wedge *_\Sigma \alpha_\Sigma) : \alpha_\Sigma \in D(v), v \in [-2, \lambda]\} & \lambda \geq -2, \\ \{0\} & \lambda < -2. \end{cases}$$

The form α_A over $(R, \infty) \times \Sigma$ given in Proposition 2.8(b) can be decomposed into self-dual 2-forms as $\alpha_A = \alpha_A^0 + \alpha'_A$, where $\alpha_A^0 \in \mathcal{K}_C(\lambda)$ and α'_A is transverse to $\mathcal{K}_C(\lambda)$ and satisfies, for some $\lambda_- < -2$,

$$|\nabla_C^j \alpha'_A| = O(r^{\max\{2\lambda-1, \lambda_-\}-j}) \text{ as } r \rightarrow \infty \text{ for all } j \in \mathbb{N}.$$

Proof: If $\lambda < -2$ then we may choose $\alpha_A^0 = 0$ and $\lambda_- = \lambda$, so suppose from now on that $\lambda \in [-2, 0)$.

Since A is coassociative, φ_0 vanishes on the graph of α_A . By [23, Proposition 4.2] and the compatibility conditions we have imposed on Υ_C , we see as in the proof of [16, Proposition 6.9] that

$$\varphi_0(\Upsilon_C(r, \sigma, \alpha_A(r, \sigma))) = d\alpha_A(r, \sigma) + P_C(r, \sigma, \alpha_A(r, \sigma), \nabla_C \alpha_A(r, \sigma)), \quad (6)$$

where

$$|\nabla_C^j P_C(r, \sigma, \alpha_A(r, \sigma), \nabla_C \alpha_A(r, \sigma))| = O(r^{2\lambda-2-j}).$$

Here we have used the fact that $r^{-1}|\alpha_A|$ and $|\nabla_C \alpha_A|$ tend to zero as $r \rightarrow \infty$.

We may view C as an AC coassociative deformation of A or vice versa, so from the discussion preceding the statement of the proposition we may decompose $\alpha_A = \alpha_A^0 + \alpha'_A$ where $\alpha_A^0 \in \mathcal{K}_C(\lambda)$ and α'_A is transverse to $\mathcal{K}_C(\lambda)$. (The

main point is that any infinitesimal AC coassociative deformation of A of rate above -2 is given by the lift of an element in $K_C(\lambda)$.) Now $d\alpha_A^0 = 0$ so

$$d\alpha'_A(r, \sigma) = -P_C(r, \sigma, \alpha_A(r, \sigma), \nabla_C \alpha_A(r, \sigma))$$

and thus $|\nabla_C^j d\alpha'_A| = O(r^{2\lambda-2-j})$ for all $j \in \mathbb{N}$. Since α'_A is transverse to $\mathcal{K}_C(\lambda)$ and $2\lambda - 2 \neq -1$, we may integrate and choose α'_A so that $|\alpha'_A| = O(r^{2\lambda-1})$. \square

Remark We see that if $\lambda < -2$ then Proposition 3.3 is irrelevant. This proposition marks the significant departure from the work in [17].

We now make some observations to understand the obstructions to the gluing procedure. If we desingularize N using tA we will obtain a smooth 4-dimensional submanifold $\tilde{N}(t)$ of M which is diffeomorphic to the disjoint union of the compact sets tK_A and K_N and the portion of the cone $(tR, \epsilon) \times \Sigma$.

Suppose we construct $\tilde{N}(t)$ so that $|\varphi|_{\tilde{N}(t)} = O(t^\eta)$ as $t \rightarrow 0$. Let $\iota_t : \Sigma \rightarrow \tilde{N}(t)$ be an inclusion of $t(R+1) \times \Sigma$ in $\tilde{N}(t)$ in the obvious way. Then

$$[\varphi|_{\tilde{N}(t)}] \cdot [\iota_t \Sigma] = \int_{\iota_t \Sigma} \varphi = O(t^{\eta+3}).$$

This product must go to zero as $t \rightarrow 0$ if we are to have any hope that $\tilde{N}(t)$ can be perturbed to a coassociative smoothing of N , so we require $\eta < -3$.

We can view the subset of $\tilde{N}(t)$ which is diffeomorphic to $(tR, \epsilon) \times \Sigma$ as the graph of a self-dual 2-form α which is at best of order $O(r^\lambda)$, since we are using A to construct $\tilde{N}(t)$. Using a similar equation to (6), naively the behaviour of $|\varphi|_{\tilde{N}(t)}$ is dominated by $|d\alpha| = O(r^{\lambda-1})$, so it would appear that we need $\lambda < -2$ so that $|\varphi|_{\tilde{N}(t)} = O(t^\eta)$ for $\eta < -3$. (This is a way to interpret how this condition arises in [17].) Hence for rates $\lambda \geq -2$ we should see obstructions to our gluing procedure, whereas for $\lambda < -2$ we should not.

However, if we can arrange α to be *closed* then, again using an equation like (6), we have that $|\varphi|_{\tilde{N}(t)}$ is now dominated by $|r^{-1}\alpha|^2$ and $|\nabla\alpha|^2$, which are of order $O(r^{2\lambda-2})$. We have thus improved our estimate drastically as we now only require $2\lambda - 2 < -3$ for our analysis to go through, which is equivalent to $\lambda < -\frac{1}{2}$. (This in part can be seen from Proposition 3.3, where now $2\lambda - 1 < -2$, so $\max\{2\lambda - 1, \lambda_-\} < -2$.)

We deduce that for rates $\lambda \in [-2, -\frac{1}{2})$, the obstructions to the desingularization arise purely from the ability to glue tA and N using a closed self-dual 2-form, which is a natural constraint in the context of coassociative geometry.

These considerations allow us to state the key condition that we require to overcome the obstructions. Notice that since α_A^0 is homogeneous it is defined on the entire cone C .

Definition 3.4 Let $\mathcal{D} \cap [-2, \lambda] = \{\lambda_1, \dots, \lambda_d\}$ with $\lambda_1 < \dots < \lambda_d$. Write α_A^0 given by Proposition 3.3 as $\alpha_A^0 = \sum_{i=1}^d \alpha_A^i$, so there are $\alpha_\Sigma^i \in D(\lambda_i)$ such that

$$\alpha_A^i(r, \sigma) = r^{\lambda_i+2} (\alpha_\Sigma^i(\sigma) + r^{-1} dr \wedge *_\Sigma \alpha_\Sigma^i(\sigma)).$$

We say that A and N satisfy the *matching condition* if there exists $\delta_0 > 0$ and for $i = 1, \dots, d$ there exists a closed self-dual form α_N^i on \hat{N} such that

$$|\nabla_C^j (\Psi_N^* \alpha_N^i(r, \sigma) - \alpha_A^i(r, \sigma))| = O(r^{\lambda_i + \delta_0 - j}) \text{ as } r \rightarrow 0 \text{ for all } j \in \mathbb{N}.$$

Effectively, this says that each infinitesimal coassociative deformation α_A^i of the cone C extends to an infinitesimal coassociative deformation α_N^i of \hat{N} so that to “leading order” α_N^i tends to α_A^i as $r \rightarrow 0$.

Remark As we shall see, the matching condition precisely allows us to define our desingularization so that over $(tR, \epsilon) \times \Sigma$ it is the graph of a self-dual 2-form whose leading order term is closed.

We now give further geometric meaning for (part of) our matching condition. Notice that if $v \neq -2$ and $\alpha_\Sigma \in D(v)$ then α_Σ is exact, whereas if $\alpha_\Sigma \in D(-2)$ then α_Σ uniquely determines a cohomology class in $H^2(\Sigma)$. Hence, there exists unique $\alpha_\Sigma^0 \in D(-2)$ such that $[\alpha_A^0] = [\alpha_\Sigma^0] \in H^2(\Sigma) \cong H^2(C)$.

In the notation of Definition 3.4, we have that $\alpha_\Sigma^0 = 0$ if $\lambda_1 > -2$ and $\alpha_\Sigma^0 = \alpha_\Sigma^1$ if $\lambda_1 = -2$. The class $[\alpha_\Sigma^0]$ is not mysterious but has a natural geometric interpretation. Recall the map j_A given by Proposition 2.6 and (5).

Proposition 3.5 *Let v be the dilation vector field on \mathbb{R}^7 and let u be the projection of v onto the normal bundle of A . We have that $dj_A(u) = 0$,*

$$[j_A(u)] = 3[\varphi_0] \in H^3(\mathbb{R}^7; A) \cong H^2(A) \quad \text{and}$$

$$j_2^A[j_A(u)] = 3[\alpha_\Sigma^0] \in H^2(\Sigma).$$

Remark Proposition 3.5 says that the cohomology class of the infinitesimal dilation deformation of A is a multiple of the class of φ_0 in $H^3(\mathbb{R}^7; A) \cong H^2(A)$, which itself projects to the class of α_Σ^0 in $H^2(\Sigma)$.

Proof: First $j_A(u) = u \lrcorner \varphi_0|_A = v \lrcorner \varphi_0|_A$ since $\varphi_0|_A = 0$. Now φ_0 is homogeneous of degree 3 so $d(v \lrcorner \varphi_0) = \mathcal{L}_v \varphi_0 = 3\varphi_0$. Hence $dj_A(u) = 3\varphi_0|_A = 0$. From this formula one also deduces that $[j_A(u)] = 3[\varphi_0] \in H^3(\mathbb{R}^7; A)$.

The dilation deformation, $A \mapsto tA$ for $t > 0$, is defined by a self-dual 2-form on A . Over C the dilation is given by $\alpha_A \mapsto t^3 \alpha_A$ from (3). Thus, the dilation

deformation is defined by $\frac{d}{dt}|_{t=1}(t^3\alpha_A) = 3\alpha_A$ over C . Hence, the corresponding infinitesimal deformation is the closed part of $3\alpha_A$ which, to leading order (that is, for order at least $O(r^{-2})$), is given by $3\alpha_A^0$. Now $j_A(u)$ also defines the infinitesimal dilation deformation, so $\Phi_{AJA}^*(u) = 3\alpha_A^0$ plus terms with order strictly less than -2 . Hence $j_2^A[j_A(u)] = 3[\alpha_A^0] = 3[\alpha_\Sigma^0]$ as claimed. \square

As for A in (5), we have an exact sequence for \hat{N} :

$$\cdots \longrightarrow H_{\text{cs}}^m(\hat{N}) \xrightarrow{\iota_m^N} H^m(\hat{N}) \xrightarrow{j_m^N} H^m(\Sigma) \xrightarrow{\partial_m^N} H_{\text{cs}}^{m+1}(\hat{N}) \longrightarrow \cdots \quad (7)$$

We can now interpret part of the matching condition in topological terms.

Proposition 3.6 *Suppose that $[\alpha_\Sigma^0] \in \text{Im } j_2^N$. There exists a closed self-dual 2-form α on \hat{N} and $\delta_0 > 0$ such that, for all $j \in \mathbb{N}$,*

$$|\nabla_C^j(\Psi_N^*\alpha(r, \sigma) - (\alpha_\Sigma^0(\sigma) + r^{-1}dr \wedge *_\Sigma \alpha_\Sigma^0(\sigma)))| = O(r^{-2+\delta_0-j}) \text{ as } r \rightarrow 0.$$

Proof: Since α_Σ^0 is a closed 2-form on C such that $[\alpha_\Sigma^0] \in \text{Im } j_2^N$, we can pull it back to the end of \hat{N} and extend it smoothly to define a closed 2-form β (c.f. [21, Proposition 5.8 & Corollary 5.9]) so that $|\nabla_C^j \Psi_N^* \beta| = O(r^{-2-j})$ as $r \rightarrow 0$ for all $j \in \mathbb{N}$. Moreover α_Σ^0 is coclosed, so since the metric on \hat{N} converges to g_C with rate $O(r^{\mu-1})$ and a dilation-invariant 3-form on C has order $O(r^{-3})$, we have that $|\nabla_C^j \Psi_N^* d*\beta| = O(r^{\mu-4-j})$ as $r \rightarrow 0$ for all $j \in \mathbb{N}$. Thus, if $\gamma = \beta + *\beta$,

$$|\nabla_C^j(\Psi_N^*\gamma(r, \sigma) - (\alpha_\Sigma^0(\sigma) + r^{-1}dr \wedge *_\Sigma \alpha_\Sigma^0(\sigma)))| = O(r^{\mu-3-j}) \text{ as } r \rightarrow 0.$$

Since $\mu - 3 > -2$ we would be done except that γ is not necessarily closed.

Now, $d\gamma$ lies in the space of exact forms which decay at rate $O(r^{\mu-4})$, so lies in the image of d acting on 2-forms which decay with rate $O(r^{\mu-3})$. As we shall see in §3.5, given $k \geq 4$ and $\delta_0 > 0$ such that $(-2, -2 + \delta_0] \cap \mathcal{D} = \emptyset$ we have that

$$d(L_{k, -2+\delta_0}^2(\Lambda^2 T^* \hat{N})) = d(L_{k, -2+\delta_0}^2(\Lambda_+^2 T^* \hat{N}))$$

(see [17, §3.4] for example for the definition of the weighted Sobolev spaces, which control the decay rate of forms near the singularity). Choosing $-2 + \delta_0 < \mu - 3$, there exists $\gamma' \in L_{k, -2+\delta_0}^2(\Lambda_+^2 T^* \hat{N})$ such that $d\gamma' = d\gamma$. Taking $\alpha = \gamma - \gamma'$ and elliptic regularity gives the result. \square

We deduce from Proposition 3.6 that part of the matching condition is purely topological; that is, we can replace the condition that a closed self-dual 2-form exists on \hat{N} asymptotic to the rate -2 part of α_A^0 with the assumption that $[\alpha_\Sigma^0]$ lies in $\text{Im } j_2^N$. This motivates the following definition for convenience.

Definition 3.7 Recall (5) and (7). We say that A and N satisfy the *topological matching condition* if

$$j_2^A[\varphi_0] \in j_2^N(H^2(\hat{N})) \subseteq H^2(\Sigma),$$

where φ_0 defines a cohomology class $[\varphi_0] \in H^3(\mathbb{R}^7; A) \cong H^2(A)$.

We have identified part of the matching condition as a topological constraint, but the remainder is analytic and still needs to be interpreted geometrically. As we remarked, homogeneous closed self-dual 2-forms on C with rates in $(-2, 0)$ always extend to A . However, this is not the case for \hat{N} , and such forms which do not extend correspond to *obstructions* to the deformation theory of \hat{N} (c.f. [16]). Therefore if the deformation theory of \hat{N} is *unobstructed*, the analytic part of matching condition will hold. We shall discuss these ideas in detail later.

The work in this subsection leads us to impose the following conditions on A and N from now on.

Conditions Assume that

- the rate λ of convergence of A to C satisfies $\lambda < -\frac{1}{2}$ and
- A and N satisfy the matching condition in Definition 3.4.

In particular, the topological matching condition in Definition 3.7 is satisfied.

From our discussion it is clear that, unless we make further assumptions or develop an even more sophisticated construction, the conditions we have imposed will be essential for our analysis to go through.

We shall see later that the condition $\lambda < -\frac{1}{2}$ allows for far more examples of conical singularities than the situation in [17] where $\lambda < -2$. One could conceivably impose further conditions on α for $\lambda \geq -\frac{1}{2}$ to ensure that $|\varphi|_{\tilde{N}(t)}$ has the required decay property for our analysis to work, but these appear to be less geometrically natural so we choose not to pursue this.

3.2 Construction

We now define our (approximately coassociative) desingularizations $\tilde{N}(t)$ of N using tA . Recall the notation of the matching condition in Definition 3.4 and suppose without loss of generality that δ_0 is small enough that

$$\mu > 1 + 2\delta_0. \tag{8}$$

Definition 3.8 Let $f_{\text{inc}} : \mathbb{R} \rightarrow [0, 1]$ be a smooth increasing function such that

$$f_{\text{inc}}(x) = \begin{cases} 0 & \text{for } x \leq 0, \\ 1 & \text{for } x \geq 1 \end{cases}$$

and $f_{\text{inc}}(x) \in (0, 1)$ for $x \in (0, 1)$. Let $\nu > 0$ be such that

$$\frac{1 - \lambda}{1 + \delta_0 - \lambda} < \nu < 1. \quad (9)$$

Choose τ sufficiently small so that

$$0 < \tau R < \frac{1}{2} \tau^\nu < \tau^\nu < \epsilon.$$

Let $\alpha_N^0(t) = \sum_{i=1}^d t^{1-\lambda_i} \alpha_N^i$ and define $\alpha_C(t)$ on $(tR, \epsilon) \times \Sigma$ by

$$\begin{aligned} \alpha_C(t)(r, \sigma) &= t^3(1 - f_{\text{inc}}(2t^{-\nu}r - 1))\alpha_A(t^{-1}r, \sigma) \\ &\quad + f_{\text{inc}}(2t^{-\nu}r - 1)(\Psi_N^* \alpha_N^0(t)(r, \sigma) + \alpha_N(r, \sigma)) \\ &= t^3 \alpha_A^0(t^{-1}r, \sigma) + t^3(1 - f_{\text{inc}}(2t^{-\nu}r - 1))\alpha'_A(t^{-1}r, \sigma) \\ &\quad + f_{\text{inc}}(2t^{-\nu}r - 1) \sum_{i=1}^d t^{1-\lambda_i} (\Psi_N^* \alpha_N^i(r, \sigma) - \alpha_A^i(r, \sigma)) \\ &\quad + f_{\text{inc}}(2t^{-\nu}r - 1)\alpha_N(r, \sigma). \end{aligned} \quad (10)$$

Observe that

$$\alpha_C(t)(r, \sigma) = \begin{cases} t^3 \alpha_A(t^{-1}r, \sigma) & r \in (tR, \frac{1}{2}t^\nu), \\ \Psi_N^* \alpha_N^0(t)(r, \sigma) + \alpha_N(r, \sigma) & r \in (t^\nu, \epsilon). \end{cases}$$

Therefore, if we let

$$\tilde{N}(t) = \chi(tK_A) \cup \Upsilon_C(\Gamma_{\alpha_C(t)}) \cup \Upsilon_N(\Gamma_{\alpha_N^0(t)|_{K_N}})$$

then, by Proposition 2.9, $\tilde{N}(t)$ is a smooth compact 4-fold so that $\tilde{N}(t) \rightarrow N$ as $t \rightarrow 0$ in the sense of currents in Geometric Measure Theory.

Remark The choice of ν in (9) will remain mysterious until late in the argument but, roughly, we need to choose ν close to 1 so that the interpolation region $r \in [\frac{1}{2}t^\nu, t^\nu]$, where φ potentially has the worst behaviour, is small as $t \rightarrow 0$.

Our aim is to solve the following problem.

Problem Deform $\tilde{N}(t)$ to a nearby coassociative 4-fold $N(t)$, so that $N(t) \rightarrow N$ as $t \rightarrow 0$ in the sense of currents.

Informally, if $|\varphi|_{\tilde{N}(t)}$ is sufficiently small we hope to make it vanish after a small perturbation. It is clear from the observations in §3.1 that the conditions we imposed precisely ensure that obstructions to this procedure can be overcome.

Our problem involves deforming the *non-coassociative* $\tilde{N}(t)$. As we have seen, infinitesimal deformations of coassociative 4-folds are defined by closed self-dual 2-forms, resulting in a deformation theory determined by solutions to an elliptic problem. To exploit this fact we want to be able to identify normal deformations to $\tilde{N}(t)$ with self-dual 2-forms, even though $\tilde{N}(t)$ is *not* coassociative. We achieve this as in [17] by defining the self-dual 2-forms with respect to a different metric on $\tilde{N}(t)$ from the induced one. This follows from an easy modification of [17, Proposition 2.9 & Lemma 4.3], since all one requires is that $\|\varphi|_{\tilde{N}(t)}\|_{C^0}$ is smaller than some universal constant, and we can in fact make this norm arbitrarily small by choosing τ sufficiently small since $\tilde{N}(t) \rightarrow N$.

Proposition 3.9 *Define*

$$j_t : \nu(\tilde{N}(t)) \rightarrow \Lambda^2 T^* \tilde{N}(t) \quad \text{by} \quad j_t : v \mapsto (v \lrcorner \varphi)|_{\tilde{N}(t)}.$$

If τ is sufficiently small, there exists a unique metric $\tilde{g}(t)$ on $\tilde{N}(t)$ such that

$$\text{Im } j_t = (\Lambda_+^2)_{\tilde{g}(t)} T^* \tilde{N}(t) \quad \text{and} \quad * \varphi|_{\tilde{N}(t)} = \text{vol}_{\tilde{g}(t)}.$$

Note From now on we shall calculate all quantities on $\tilde{N}(t)$ with respect to the metric $\tilde{g}(t)$ given in Proposition 3.9, unless stated otherwise, and we shall use the notation of Definition 3.8.

3.3 Weighted spaces

We wish to define spaces of forms on $\tilde{N}(t)$ whose behaviour on the piece we have glued into N is controlled, since this is where the geometry is degenerating. We achieve this using Banach spaces with weighted norms, as discussed in detail in [25]. To define these spaces we need an appropriate radius function.

Definition 3.10 Recall Definition 3.8 and let R', ϵ' be constants so that

$$\tau R < \tau R' < \frac{1}{2} \tau^\nu < \tau^\nu < \epsilon' < \epsilon.$$

We define a radius function $\rho_t : \tilde{N}(t) \rightarrow [tR, \epsilon]$ as a smooth map such that

$$\rho_t(x) = \begin{cases} tR & x \in \chi(tK_A), \\ \epsilon & x \in \Upsilon_N(\Gamma_{\alpha_N^0(t)|_{K_N}}), \end{cases}$$

and ρ_t on $\Upsilon_C(\Gamma_{\alpha_C(t)})$ is a strictly increasing function of r such that

$$\rho_t(\Upsilon_C(r, \sigma, \alpha_C(t)(r, \sigma))) = r \quad \text{for } r \in [tR', \epsilon'].$$

In other words, ρ_t is a small perturbation of the piecewise smooth function which is constant on $\chi(tK_A)$ and $\Upsilon_N(\Gamma_{\alpha_N^0(t)|_{K_N}})$ and equal to r on $\Upsilon_C(\Gamma_{\alpha_C(t)})$. The existence of such a function ρ_t on $\tilde{N}(t)$ is clear.

Definition 3.11 Let $k \in \mathbb{N}$, $p \geq 1$ and $v \in \mathbb{R}$.

Define $C_{v,t}^k(\Lambda^m T^* \tilde{N}(t))$ to be the space $C^k(\Lambda^m T^* \tilde{N}(t))$ with the norm

$$\|\xi\|_{C_{v,t}^k} = \sum_{j=0}^k \sup_{\tilde{N}(t)} |\rho_t^{j-v} \nabla^j \xi|.$$

Let $L_{k,v,t}^p(\Lambda^m T^* \tilde{N}(t))$ be the space $L_k^p(\Lambda^m T^* \tilde{N}(t))$ with the norm

$$\|\xi\|_{L_{k,v,t}^p} = \sum_{j=0}^k \left(\int_{\tilde{N}(t)} |\rho_t^{j-v} \nabla^j \xi|^p \rho_t^{-4} \mathrm{dvol}_{\tilde{g}(t)} \right)^{\frac{1}{p}}.$$

These weighted spaces are Banach spaces since the norms are equivalent to the usual norms for each fixed t .

These spaces have the nice feature, unlike their “unweighted” counterparts, that they are equivariant under dilations in t . Thus, with respect to these weighted norms, we can understand the behaviour of quantities as $t \rightarrow 0$, which is crucial for our analysis. For a detailed discussion of these issues see [25].

On CS and AC submanifolds we can define Sobolev, C^k and Hölder spaces of forms, denoted $L_{k,v}^p$, C_v^k and $C_v^{k,a}$, which depend on a weight $v \in \mathbb{R}$. Informally, these Banach spaces consist of forms whose restriction to any compact set lies in the usual Sobolev, C^k or Hölder space, but which also have controlled rate of decay on the ends determined by v . This is achieved using weighted norms as in Definition 3.11, replacing ρ_t by an appropriate radius function – we refer the interested reader to [17, §3.3] or [25] for details. We point out that $L_{0,-2}^2 = L^2$.

3.4 Topology

Clearly we need to know $b_+^2(\tilde{N}(t))$ to understand deformations of $\tilde{N}(t)$. This is a topological invariant which we can determine using the topology of \hat{N} and A .

Theorem 3.12 *Using the notation of Definition 2.11, (5) and (7), we have that*

$$b_+^2(\tilde{N}(t)) = b_+^2(A) + b_+^2(\hat{N}) + \dim(\mathrm{Im} j_2^A \cap \mathrm{Im} j_2^N). \quad (11)$$

Proof: We can clearly choose a pair of open subsets of $\tilde{N}(t)$, diffeomorphic to A and \hat{N} , which cover $\tilde{N}(t)$ and whose intersection is diffeomorphic to C . By Mayer–Vietoris we then have the following exact sequence:

$$\cdots \longrightarrow H^m(\tilde{N}(t)) \xrightarrow{\tilde{l}_m} H^m(A) \oplus H^m(\hat{N}) \xrightarrow{\tilde{j}_m} H^m(\Sigma) \xrightarrow{\tilde{\partial}_m} H^{m+1}(\tilde{N}(t)) \longrightarrow \cdots \quad (12)$$

Since (12) is exact we have that

$$b^2(\tilde{N}(t)) = \dim \operatorname{Ker} \tilde{l}_2 + \dim \operatorname{Im} \tilde{l}_2 = \dim \operatorname{Im} \tilde{\partial}_1 + \dim \operatorname{Ker} \tilde{j}_2. \quad (13)$$

We first calculate $\dim \operatorname{Im} \tilde{\partial}_1$. Using (5) we see that $\dim \operatorname{Im} \partial_1^A = \dim \operatorname{Im} j_2^A$ and the same result holds for N by (7). The fact that these spaces have the same dimension is a consequence of Poincaré duality, which further allows us to construct an isomorphism between them. Now $\tilde{\partial}_1$ is defined so that $\tilde{\partial}_1[\alpha_\Sigma]$ can be simultaneously viewed as $\partial_1^A[\alpha_\Sigma] \in H_{\text{cs}}^1(A)$ and $\partial_1^N[\alpha_\Sigma] \in H_{\text{cs}}^1(\hat{N})$. Hence, $\operatorname{Im} \tilde{\partial}_1$ is dual to the intersection of $\operatorname{Im} j_2^A$ and $\operatorname{Im} j_2^N$. We conclude that

$$\dim \operatorname{Im} \tilde{\partial}_1 = \dim(\operatorname{Im} j_2^A \cap \operatorname{Im} j_2^N). \quad (14)$$

We now determine $\dim \operatorname{Ker} \tilde{j}_2$. By definition, $\tilde{j}_2 = j_2^A - j_2^N$, so may also calculate, recalling Definition 2.11,

$$\begin{aligned} \dim \operatorname{Ker} \tilde{j}_2 &= \dim \operatorname{Ker} j_2^A + \dim \operatorname{Ker} j_2^N + \dim(\operatorname{Im} j_2^A \cap \operatorname{Im} j_2^N) \\ &= \dim \mathcal{J}(A) + \dim \mathcal{J}(\hat{N}) + \dim(\operatorname{Im} j_2^A \cap \operatorname{Im} j_2^N). \end{aligned} \quad (15)$$

We deduce from (13), (14) and (15) that

$$b^2(\tilde{N}(t)) = \dim \mathcal{J}(A) + \dim \mathcal{J}(\hat{N}) + 2 \dim(\operatorname{Im} j_2^A \cap \operatorname{Im} j_2^N).$$

There is no obstruction to elements of $\operatorname{Im} j_2^N \subseteq H^2(\Sigma)$ lifting to closed 2-forms on \hat{N} which are either self-dual or anti-self-dual by Proposition 3.6, and a similar result holds on A (as noted after Theorem 3.2). We conclude that

$$b_\pm^2(\tilde{N}(t)) = b_\pm^2(A) + b_\pm^2(\hat{N}) + \dim(\operatorname{Im} j_2^A \cap \operatorname{Im} j_2^N)$$

and deduce (11). \square

Remarks From the proof of Theorem 3.12 we deduce a special case of the Novikov additivity theorem, namely that

$$b_+^2(\tilde{N}(t)) - b_-^2(\tilde{N}(t)) = b_+^2(A) - b_-^2(A) + b_+^2(\hat{N}) - b_-^2(\hat{N}).$$

This is unsurprising since our proof essentially follows the argument in [1] for proving the Novikov additivity theorem.

3.5 Stability

Theorem 3.2 allows us to conclude that infinitesimal deformations of C which are homogeneous of rate $(-2, 0)$ always extend to genuine deformations of A , so the deformation theory of A is unobstructed. This unobstructedness follows from the fact that, for $\lambda_+ \geq \lambda$ such that $\lambda_+ \in (-2, 0) \setminus \mathcal{D}$,

$$d(L_{4,\lambda_+}^2(\Lambda^2 T^* A)) = d(L_{4,\lambda_+}^2(\Lambda_+^2 T^* A)).$$

In contrast on \hat{N} we find that the images of d on 2-forms and self-dual 2-forms in $L_{4,\mu}^2$ can differ.

Definition 3.13 By the work in [16, §6] we have that if $\mu_+ \notin \mathcal{D}$ then there exists a finite-dimensional subspace $\mathcal{O}(N, \mu_+)$ of $L_{3,\mu_+-1}^2(\Lambda^3 T^* \hat{N})$ such that

$$d(L_{4,\mu_+}^2(\Lambda^2 T^* \hat{N})) = d(L_{4,\mu_+}^2(\Lambda_+^2 T^* \hat{N})) \oplus \mathcal{O}(N, \mu_+).$$

We call $\mathcal{O}(N, \mu)$ the *obstruction space* since one of the main results in [16] states that if $\mathcal{O}(N, \mu) = \{0\}$ then N has a smooth moduli space of deformations as a CS coassociative 4-fold; that is, its deformation theory is *unobstructed*.

We shall now show that the obstruction space corresponds to closed self-dual 2-forms on C which do *not* lift to \hat{N} . From our matching condition in Definition 3.4, we see that these are exactly the sort of obstructions we need to overcome in order to solve our gluing problem. This allows us to draw a direct link between obstructions to the smoothing of N and obstructions to deformations of N .

We begin with the following result from [16].

Proposition 3.14 Recall Definition 3.1, let μ_0 be the least element of $((-2, 0) \cap \mathcal{D}) \cup \{0\}$ and let $\mu_- \in (-2, \mu_0)$. If $\mu_+ \in (-2, 0) \setminus \mathcal{D}$ with $\mu_+ > \mu_-$ then the dimension of the kernel of d in $L_{4,\mu_+}^2(\Lambda_+^2 T^* \hat{N})$ is

$$b_+^2(\hat{N}) + \dim \text{Ker}(d_+^* + d)_{\mu_+} - \dim \text{Ker}(d_+^* + d)_{\mu_-} - \sum_{v \in (-2, \mu_+)} d_{\mathcal{D}}(v),$$

where $(d_+^* + d)_v$ acts on $L_{4,-3-v}^2(\Lambda^3 T^* \hat{N})$.

The reason for the appearance of $\text{Ker}(d_+^* + d)_v$ is that it is isomorphic to the cokernel of the map

$$(d_+ + d^*)_v : L_{4,v}^2(\Lambda_+^2 T^* \hat{N} \oplus \Lambda^4 T^* \hat{N}) \rightarrow L_{3,v-1}^2(\Lambda^3 T^* \hat{N}). \quad (16)$$

We need to relate $\text{Ker}(d_+^* + d)_v$ to the space of closed and coclosed 3-forms, since $\mathcal{O}(N, \mu_+)$ is isomorphic to the subspace of $\text{Ker}(d_+^* + d)_{\mu_+}$ which is orthogonal to these forms by the work in [16]. We begin with the following observation.

Lemma 3.15 *If $\gamma \in L^2_{4,-1}(\Lambda^3 T^* \hat{N})$ and $d_+^* \gamma = 0$ then $d^* \gamma = 0$.*

Proof: Since $d^* \gamma \in L^2_{3,-2} \hookrightarrow L^2$ we can calculate

$$\|d^* \gamma\|_{L^2} = - \int_{\hat{N}} d^* \gamma \wedge d^* \gamma = \int_{\hat{N}} d(*\gamma \wedge d * \gamma) = 0,$$

using the fact that $d^* \gamma$ is anti-self-dual and the decay properties of γ to ensure the integration by parts is valid. \square

It follows from Lemma 3.15 that, for $v \leq -2$, $\text{Ker}(d_+^* + d)_v$ is equal to

$$\mathcal{H}_v^3(\hat{N}) = \{\gamma \in L^2_{4,-3-v}(\Lambda^3 T^* \hat{N}) : d\gamma = d^* \gamma = 0\}. \quad (17)$$

Thus, the obstruction space $\mathcal{O}(N, v) = \{0\}$ if $v \leq -2$. However, we need to calculate $\dim \mathcal{O}(N, \mu_+)$ for $\mu_+ \in (-2, 0)$ so we need to compare $\text{Ker}(d_+^* + d)_{\mu_+}$ and $\mathcal{H}_{\mu_+}^3$ for these rates. This is made easier by the following fact.

Proposition 3.16 *The space $\mathcal{H}_{\mu_+}^3(\hat{N})$ in (17) is the same for all $\mu_+ \in (-2, 0)$.*

Proof: By the work in [15], changes in $\mathcal{H}_{\mu_+}^3(\hat{N})$ are governed by homogeneous closed and coclosed 3-forms γ_∞ on C of rate $-3 - v$ for $v \in (-2, 0)$. We write

$$\gamma_\infty = r^{-3-v}(r^3 \beta_\Sigma + r^2 dr \wedge \alpha_\Sigma) \quad (18)$$

for forms $\alpha_\Sigma, \beta_\Sigma$ on Σ . The condition that $d\gamma_\infty = d^* \gamma_\infty = 0$ is equivalent to

$$d\alpha_\Sigma = -v\beta_\Sigma, \quad d*\Sigma \alpha_\Sigma = 0 \quad \text{and} \quad d*\Sigma \beta_\Sigma = (v+2)*\Sigma \alpha_\Sigma. \quad (19)$$

We deduce that

$$\Delta_\Sigma \beta_\Sigma = v(v+2)\beta_\Sigma.$$

Thus $\beta_\Sigma = 0$ if $v \in (-2, 0)$ and hence by (19) $\alpha_\Sigma = 0$ as well. \square

As previously mentioned, the work in [16] shows that

$$\dim \text{Ker}(d_+^* + d)_v = \dim \mathcal{H}_v^3(\hat{N}) + \dim \mathcal{O}(N, v).$$

Using the notation of Proposition 3.14, applying Proposition 3.16 gives:

$$\dim \text{Ker}(d_+^* + d)_{\mu_+} - \dim \text{Ker}(d_+^* + d)_{\mu_-} = \dim \mathcal{O}(N, \mu_+) - \dim \mathcal{O}(N, \mu_-).$$

As we saw, $\mathcal{O}(N, v) = \{0\}$ for $v \leq -2$. Since $\mathcal{O}(N, \mu_-) = \mathcal{O}(N, -2 + \varepsilon)$ for arbitrarily small $\varepsilon > 0$, to finish we need to calculate how the obstruction space changes as the rate crosses -2 . The next lemma states that it does not change.

Lemma 3.17 *Using the notation of Proposition 3.14 and (17),*

$$\text{Ker}(\text{d}_+^* + \text{d})_{\mu_-} = \mathcal{H}_{\mu_-}^3(\hat{N}) \quad \text{or, equivalently,} \quad \mathcal{O}(N, \mu_-) = \{0\}.$$

Proof: A form γ adds to $\text{Ker}(\text{d}_+^* + \text{d})_{\mu_-}$ at rate -2 if and only if it is asymptotic to a 3-form γ_∞ on C of rate $-3 - (-2) = -1$ which satisfies $\text{d}\gamma_\infty = \text{d}_+^* \gamma_\infty = 0$ by the work in [15]. Hence, if we abuse notation and identify γ_∞ with its pull-back to \hat{N} , then $\gamma - \gamma_\infty \in \text{Ker}(\text{d}_+^* + \text{d})_{-2}$. Since $\text{Ker}(\text{d}_+^* + \text{d})_{-2} = \mathcal{H}_{-2}^3(\hat{N})$ by Lemma 3.15, $\gamma - \gamma_\infty$ is closed and coclosed, so if we show that $\text{d}\gamma_\infty = \text{d}_+^* \gamma_\infty = 0$ then $\gamma \in \mathcal{H}_{\mu_-}^3(\hat{N})$ as we desire.

Writing γ_∞ as in (18) for $v = -2$, we see that α_Σ and β_Σ satisfy

$$\text{d}\alpha_\Sigma = 2\beta_\Sigma \quad \text{and} \quad *_\Sigma \text{d} *_\Sigma \alpha_\Sigma + \text{d} *_\Sigma \beta_\Sigma = 0.$$

Taking d_Σ^* of the second equation, we deduce that β_Σ is harmonic and exact, so must be zero. Thus α_Σ is a closed and coclosed 2-form. By (19), this means that $\text{d}\gamma_\infty = \text{d}_+^* \gamma_\infty = 0$ as required. \square

Combining the results in this section we deduce the following.

Proposition 3.18 *If $\mu_+ \in (-2, 0) \setminus \mathcal{D}$ the dimension of the kernel of d in $L_{4, \mu_+}^2(\Lambda_+^2 T^* \hat{N})$ is*

$$b_+^2(\hat{N}) + \dim \mathcal{O}(N, \mu_+) - \sum_{v \in (-2, \mu_+)} d_{\mathcal{D}}(v).$$

Proposition 3.18 allows us to conclude that every homogeneous closed self-dual 2-form on C of rate $v \in (-2, \mu_+)$ lifts to a closed self-dual 2-form on \hat{N} if and only if $\mathcal{O}(N, \mu_+) = \{0\}$. In particular, α_A^0 given in Proposition 3.3 will lift to \hat{N} as in the matching condition in Definition 3.4 if $j_2^A[\alpha_A^0] \in j_2^N(H^2(\hat{N}))$ and $\mathcal{O}(N, \lambda) = \{0\}$. We can therefore check that the matching condition is satisfied purely using a topological criterion and the spectrum of the curl operator on Σ . The analytic part of the matching condition therefore relates to the work in [20] on *stability* of coassociative conical singularities.

We now recall the notion of *stability index* for a coassociative cone from [20].

Definition 3.19 Let \mathcal{C} denote a deformation family of coassociative cones in \mathbb{R}^7 containing C which is closed under the action of translations and G_2 transformations. The \mathcal{C} -*stability index* of C is

$$\text{ind}_{\mathcal{C}}(C) = \sum_{v \in (-1, 1]} d_{\mathcal{D}}(v) - \dim \mathcal{C}.$$

If the family \mathcal{C} consists solely of the $G_2 \ltimes \mathbb{R}^7$ transformations of C , we simply write $\text{ind}_{\mathcal{C}}(C) = \text{ind}(C)$ and call $\text{ind}(C)$ the *stability index* of C .

Since translations of C trivially provide coassociative deformations of C of order $O(1)$, they define homogeneous closed self-dual 2-forms on C of rate 0, and thus $d_{\mathcal{D}}(0)$ is at least equal to the dimension of the space of translations of C . Moreover, $d_{\mathcal{D}}(1)$ is equal to the dimension of infinitesimal coassociative conical deformations of C . Overall, we have that $\text{ind}_{\mathcal{C}}(C) \geq 0$ and equals zero if and only if $\dim \mathcal{C} = d_{\mathcal{D}}(0) + d_{\mathcal{D}}(1)$ and $d_{\mathcal{D}}(v) = 0$ for all $v \in (-1, 1) \setminus \{0\}$.

We say that C is \mathcal{C} -stable (or *stable*) if $\text{ind}_{\mathcal{C}}(C) = 0$ (or $\text{ind}(C) = 0$).

The \mathcal{C} -stability index is a non-negative integer invariant and it follows from [20, Proposition 4.11] that the deformation theory of N as a CS coassociative 4-fold, where we allow the singularity to move in M and the cone at the singularity to deform in \mathcal{C} , is unobstructed if $\text{ind}_{\mathcal{C}}(C) = 0$. In particular, $\dim \mathcal{O}(N, \mu_+) = 0$ for all $\mu_+ \in (-2, 0)$ if $\text{ind}_{\mathcal{C}}(C) = 0$. We thus conclude with the following.

Proposition 3.20 *Recall Definitions 3.4, 3.7, 3.13 and 3.19.*

- (a) *If the topological matching condition holds and $\mathcal{O}(N, \lambda) = \{0\}$, then the matching condition between A and N holds.*
- (b) *If C is \mathcal{C} -stable for some deformation family \mathcal{C} , then the matching condition between A and N is equivalent to the topological matching condition.*

Remark Proposition 1.1 follows from Proposition 3.20(b).

4 Desingularization: analysis

In this section we apply analytic techniques to prove our main result (Theorem 1.2). We begin by deriving a key Sobolev embedding inequality, which involves the construction of an “approximate kernel” for the exterior derivative on self-dual 2-forms on our glued manifold. We then view our desingularization problem as a fixed point problem for a certain map, so we show that this map is a contraction using further analytic estimates. To derive the embedding inequality and the estimates we shall make crucial use of the geometric preliminaries of §3.

For the whole of this section we let δ satisfy

$$0 \leq \delta < \max\{-(1 + 2\lambda), 1\} \quad (20)$$

and be such that $[-2 - \delta, -2 + \delta] \cap \mathcal{D} = \{-2\}$ if $-2 \in \mathcal{D}$ and let $\delta = 0$ if $-2 \notin \mathcal{D}$. This is possible by our assumption that $\lambda < -\frac{1}{2}$ and the properties of \mathcal{D} .

4.1 The approximate kernel

Here we obtain our Sobolev embedding inequality, closely following the work in [17] with improvements in light of Pacini's work in [25]. The result is a bound (depending on t in an explicit way) for the norm of self-dual 2-forms α on $\tilde{N}(t)$, transverse to the closed forms, by the norm of $d\alpha$. Since we have scale-invariant Sobolev embedding inequalities on A and \hat{N} , the idea is to use closed self-dual 2-forms on A and \hat{N} to build a subspace of the self-dual 2-forms on $\tilde{N}(t)$ which "approximates" the kernel of the exterior derivative on self-dual 2-forms.

If the closed self-dual 2-forms on A and \hat{N} decay sufficiently fast on the ends, we can simply cut them off and approximate them using a compactly supported 2-form which can easily be viewed as an approximate kernel form on $\tilde{N}(t)$. However, at the critical decay rate, namely at the L^2 growth rate -2 , this cut off procedure will not work and so one needs to define approximate kernel forms by interpolating between L^2 kernel forms on A and L^2 kernel forms on \hat{N} , when this is possible. Using the topological calculations in §3, we find that these forms define an approximate kernel of equal dimension to the actual kernel.

However, the interpolation between L^2 kernel forms on A and \hat{N} is not always possible, and one can detect this topologically by the work in §3.1. When this occurs, we have closed self-dual 2-forms on A which do *not* define approximate kernel forms on $\tilde{N}(t)$ and so give potential obstructions. These forms will cause the Sobolev embedding constant to blow up as $t \rightarrow 0$ but because we can identify the forms explicitly we can determine the rate at which the blow up occurs.

Proposition 4.1 *Let \mathcal{K}_\pm^A be the (finite-dimensional) kernel of*

$$d : L_{4,-2\pm\delta}^2(\Lambda_+^2 T^* A) \rightarrow L_{3,-3\pm\delta}^2(\Lambda^3 T^* A)$$

and let \mathcal{K}_0^A be such that

$$\mathcal{K}_+^A = \mathcal{K}_-^A \oplus \mathcal{K}_0^A.$$

There is a subspace $\mathcal{K}_{\text{ap}}^A \subseteq C_{\text{cs}}^\infty(\Lambda_+^2 T^ A)$, L^2 -orthogonal to \mathcal{K}_0^A with $\dim \mathcal{K}_{\text{ap}}^A = \dim \mathcal{K}_-^A$, and a constant $C(A) > 0$ such that if $\alpha \in L_{4,-2+\delta}^2(\Lambda_+^2 T^* A)$ satisfies $\langle \alpha, \beta \rangle_{L^2} = 0$ for all $\beta \in \mathcal{K}_{\text{ap}}^A \oplus \mathcal{K}_0^A$ then*

$$\|\alpha\|_{L_{4,-2+\delta}^2} \leq C(A) \|d\alpha\|_{L_{3,-3+\delta}^2}. \quad (21)$$

Moreover, this estimate holds for the same constant $C(A)$ on tA for all $t > 0$.

Proof: The map

$$d_+ + d^* : L_{4,-2\pm\delta}^2(\Lambda_+^2 T^* A \oplus \Lambda^4 T^* A) \rightarrow L_{3,-3\pm\delta}^2(\Lambda^3 T^* A) \quad (22)$$

is elliptic and Fredholm by choice of δ (c.f. [18, Proposition 5.4] and the remarks preceding the statement), and therefore has a finite-dimensional kernel of smooth forms by elliptic regularity. Since \mathcal{K}_\pm^A is contained in this kernel it is also necessarily finite-dimensional and consists of smooth forms.

We can cut off the forms in \mathcal{K}_-^A appropriately at infinity to define a space $\mathcal{K}_{\text{ap}}^A$ of compactly supported self-dual 2-forms on A , L^2 -orthogonal to \mathcal{K}_0^A , so that if $\alpha \in \mathcal{K}_-^A$ is L^2 -orthogonal to $\mathcal{K}_{\text{ap}}^A$, then $\alpha = 0$. (This is a manifestation of the fact that, by definition, C_{cs}^∞ is dense in $L_{4,-2-\delta}^2$.) In other words, we can ensure that the L^2 -orthogonal complement of $\mathcal{K}_{\text{ap}}^A$ in $L_{4,-2+\delta}^2$ is transverse to \mathcal{K}_-^A and contains \mathcal{K}_0^A . The theory of elliptic operators on weighted Sobolev spaces as in [22] applied to (22) allows us to deduce the existence of the constant $C(A)$ using standard techniques.

By definition of the weighted norm, if β is an m -form on A then $\|\beta\|_{L_{k,v}^2}$ scales with order t^{-v-m} under dilation by t . Therefore both sides of (21) scale by the same factor under dilation and thus we can choose $C(A)$ independent of t . (See [25] for a detailed discussion of the scaling properties of weighted Sobolev norms.) \square

We can also prove the following analogue of Proposition 4.1 in a similar (easier) manner which we omit.

Proposition 4.2 *Let \mathcal{K}^N be the (finite-dimensional) kernel of*

$$d : L_{4,-2+\delta}^2(\Lambda_+^2 T^* \hat{N}) \rightarrow L_{3,-3+\delta}^2(\Lambda^3 T^* \hat{N}).$$

There is a subspace $\mathcal{K}_{\text{ap}}^N \subseteq C_{\text{cs}}^\infty(\Lambda_+^2 T^ \hat{N})$, with $\dim \mathcal{K}_{\text{ap}}^N = \dim \mathcal{K}^N$, and a constant $C(N) > 0$ such that if $\alpha \in L_{4,-2+\delta}^2(\Lambda_+^2 T^* \hat{N})$ satisfies $\langle \alpha, \beta \rangle_{L^2} = 0$ for all $\beta \in \mathcal{K}_{\text{ap}}^N$ then*

$$\|\alpha\|_{L_{4,-2+\delta}^2} \leq C(N) \|d\alpha\|_{L_{3,-3+\delta}^2}.$$

We now wish to define our approximate kernel. We begin with $\mathcal{K}_{\text{ap}}^A$ and define a diffeomorphism

$$\Psi_{A,t} : t\hat{A}(t) = tK_A \cup t\Phi_A((R, t^{-1}\epsilon) \times \Sigma) \rightarrow \chi(tK_A) \cup \Upsilon_C(\Gamma_{\alpha_C(t)})$$

by

$$\Psi_{A,t}(x) = \begin{cases} \chi(x) & x \in tK_A, \\ \Upsilon_C(r, \sigma, \alpha_C(t)(r, \sigma)) & x = t\Phi_A(t^{-1}r, \sigma). \end{cases} \quad (23)$$

If τ is sufficiently small we may identify the metrics, hence the self-dual 2-forms, on $t\hat{A}(t)$ and $\Psi_{A,t}(t\hat{A}(t))$. This allows us to view the open subset $t\hat{A}(t)$ of tA as a subset of the desingularization $\tilde{N}(t)$.

Let $\beta_1^A, \dots, \beta_{m_A}^A$ be a basis for $\mathcal{K}_{\text{ap}}^A$. Since $\nu < 1$, we can choose τ so that

$$\text{supp } \beta_i^A \subseteq K_A \sqcup \Phi_A((R, \frac{1}{2}t^{\nu-1}) \times \Sigma) \subseteq \hat{A}(t)$$

for $i = 1, \dots, m_A$, so we may identify the β_i^A with forms on $t\hat{A}(t)$ in the obvious manner, which we denote by the same symbols. Using $\Psi_{A,t}$ we can define for each β_i^A on $t\hat{A}(t)$ a corresponding self-dual 2-form ξ_i^A on $\tilde{N}(t)$ which vanishes outside $\Psi_{A,t}(\text{supp } \beta_i^A)$.

Definition 4.3 Let $\tilde{\mathcal{K}}_{\text{ap}}^A(t) = \text{Span}\{\xi_1^A, \dots, \xi_{m_A}^A\}$.

We now deal with $\mathcal{K}_{\text{ap}}^N$ and define a diffeomorphism

$$\Psi_{N,t} : \hat{N}(t) = \Psi_N((tR, \epsilon) \times \Sigma) \cup K_N \rightarrow \Upsilon_C(\Gamma_{\alpha_C(t)}) \cup \Upsilon_N(\Gamma_{\alpha_0^N(t)|_{K_N}})$$

by

$$\Psi_{N,t}(x) = \begin{cases} \Upsilon_C(r, \sigma, \alpha_C(t)(r, \sigma)) & x = \Psi_N(r, \sigma), \\ \Upsilon_N(x, \alpha_0^N(t)(x)) & x \in K_N. \end{cases} \quad (24)$$

For τ sufficiently small we can identify the metrics, hence the self-dual 2-forms, on $\hat{N}(t)$ and $\Psi_{N,t}(\hat{N}(t))$. As above, this allows us to view the open subset $\hat{N}(t)$ of \hat{N} as a subset of $\tilde{N}(t)$.

Let $\beta_1^N, \dots, \beta_{m_N}^N$ be a basis for $\mathcal{K}_{\text{ap}}^N$. Since $\nu > 0$ we can ensure, by making τ smaller if necessary, that

$$\text{supp } \beta_i^N \subseteq K_N \cup \Psi_N((t^\nu, \epsilon) \times \Sigma) \subseteq \hat{N}(t)$$

for all i . Using $\Psi_{N,t}$ we can then define for each β_i^N a corresponding self-dual 2-form ξ_i^N on $\tilde{N}(t)$ which vanishes outside $\Psi_{N,t}(\text{supp } \beta_i^N)$.

Definition 4.4 Let $\tilde{\mathcal{K}}_{\text{ap}}^N(t) = \text{Span}\{\xi_1^N, \dots, \xi_{m_N}^N\}$.

It will be important to identify the closed self-dual 2-forms on A which extend to \hat{N} and those which do not, as we saw from our discussion of the obstructions to the gluing problem in §3.1. Motivated by Proposition 3.6 we can split \mathcal{K}_0^A in the following useful way.

Definition 4.5 Recall (5) and (7). Let

$$\mathcal{K}^{\mathcal{I}} = \{\alpha \in \mathcal{K}_0^A : j_2^A[\alpha] \in \text{Im } j_2^N\}$$

and let $\mathcal{K}^{\mathcal{O}}$ be such that $\mathcal{K}_0^A = \mathcal{K}^{\mathcal{I}} \oplus \mathcal{K}^{\mathcal{O}}$.

The notation \mathcal{I} and \mathcal{O} reflects the fact that elements in $\mathcal{K}^{\mathcal{I}}$ extend to infinitesimal deformations of \hat{N} and hence $\tilde{N}(t)$, whereas $\mathcal{K}^{\mathcal{O}}$ gives potential obstructions. Let $\beta_1^{\mathcal{I}}, \dots, \beta_{m_{\mathcal{I}}}^{\mathcal{I}}$ and $\beta_1^{\mathcal{O}}, \dots, \beta_{m_{\mathcal{O}}}^{\mathcal{O}}$ form bases of $\mathcal{K}^{\mathcal{I}}$ and $\mathcal{K}^{\mathcal{O}}$. Since

every element of $\text{Im } j_2^A$ lifts to a form in \mathcal{K}_0^A by the work in [18], we have that

$$\dim \mathcal{K}^{\mathcal{I}} = \dim(\text{Im } j_2^A \cap \text{Im } j_2^N). \quad (25)$$

We shall now define a part of the approximate kernel using $\mathcal{K}^{\mathcal{I}}$ and we explain the idea. Since each $\beta \in \mathcal{K}^{\mathcal{I}}$ is asymptotic to a closed self-dual 2-form β^C on C whose cohomology class lies in the image of $j_2^N : H^2(\hat{N}) \rightarrow H^2(\Sigma) \cong H^2(C)$, we can find a closed self-dual 2-form γ on \hat{N} which is also asymptotic to β^C . We then interpolate between β and γ to define a self-dual 2-form on $\tilde{N}(t)$ which is “almost” closed, i.e. it is closed except on some small compact set.

By the general theory in [15], as discussed in the particular case of interest in [18], for each $i = 1, \dots, m_{\mathcal{I}}$ there exists a closed self-dual 2-form β_i^C on C , which is homogeneous of order $O(r^{-2})$, such that, for some $\epsilon_0 > 0$,

$$|\nabla_C^j (\Phi_A^* \beta_i^{\mathcal{I}}(r, \sigma) - \beta_i^C(r, \sigma))| = O(r^{-2-\epsilon_0-j}) \quad \text{as } r \rightarrow \infty \text{ for all } j \in \mathbb{N}.$$

By Proposition 3.6, there exist closed self-dual 2-forms $\gamma_i^{\mathcal{I}}$ on \hat{N} so that, for some other $\epsilon_0 > 0$,

$$|\nabla_C^j (\Psi_N^* \gamma_i^{\mathcal{I}}(r, \sigma) - \beta_i^C(r, \sigma))| = O(r^{-2+\epsilon_0-j}) \quad \text{as } r \rightarrow 0 \text{ for all } j \in \mathbb{N}.$$

Define a diffeomorphism $\Psi_{C,t} : (tR, \epsilon) \times \Sigma \rightarrow \Upsilon_C(\Gamma_{\alpha_C(t)})$ by

$$\Psi_{C,t}(r, \sigma) = \Upsilon_C(r, \sigma, \alpha_C(t)(r, \sigma)). \quad (26)$$

We can now define a self-dual 2-form $\xi_i^{\mathcal{I}}$ on $\tilde{N}(t)$, for $i = 1, \dots, m_{\mathcal{I}}$, so that on $\chi(tK_A)$ it equals $\beta_i^{\mathcal{I}}$ (using the identification $\Psi_{A,t}$), on $\Upsilon_N(\Gamma_{t\alpha_N|_{K_N}})$ it equals $\gamma_i^{\mathcal{I}}$ (using the identification $\Psi_{N,t}$), and on $\Upsilon_C(\Gamma_{\alpha_C(t)})$ it interpolates between these definitions in the following way:

$$\Psi_{C,t}^* \xi_i^{\mathcal{I}}(r, \sigma) = (1 - f_{\text{inc}}(2t^{-\nu}r - 1)) \Psi_{A,t}^* \beta_i^{\mathcal{I}}(r, \sigma) + f_{\text{inc}}(2t^{-\nu}r - 1) \Psi_{N,t}^* \gamma_i^{\mathcal{I}}(r, \sigma),$$

where f_{inc} is given in Definition 3.8. Notice that $\xi_i^{\mathcal{I}}$ is closed except on the region where tA is connected to \hat{N} to form $\tilde{N}(t)$.

We now turn to $\beta \in \mathcal{K}^{\mathcal{O}}$ which do *not* define approximate kernel forms. We cut off β to define a self-dual 2-form ξ on $\tilde{N}(t)$ which vanishes on $\hat{N}(t)$ and is closed except on a small compact set so that, as $t \rightarrow 0$, ξ converges back to β on A , after re-scaling. Although ξ will again be “almost” closed, this time there is no corresponding closed self-dual 2-form on $\tilde{N}(t)$ which it approximates. Moreover, since ξ converges to a non-trivial closed form as $t \rightarrow 0$ it is clear that ξ will contribute to the blow up of the Sobolev embedding constant as $t \rightarrow 0$.

Define $f^\mathcal{O} : A \rightarrow [0, 1]$ to be a smooth function such that

$$f^\mathcal{O}(x) = 1 \text{ for } x \in K_A, \quad \text{supp } df^\mathcal{O} \subseteq \Phi_A((\tfrac{1}{2}t^{\nu-1}, t^{\nu-1}) \times \Sigma)$$

and $f^\mathcal{O}$ is decreasing in r on $\text{Im } \Phi_A$. For $i = 1, \dots, m_\mathcal{O}$ there exists a smooth self-dual 2-form $\xi_i^\mathcal{O}$ on $\tilde{N}(t)$ such that

$$\Psi_{A,t}^*(\xi_i^\mathcal{O}) = \beta_i^\mathcal{O} \text{ on } tK_A \cup t\Phi_A((R, \tfrac{1}{2}t^{\nu-1}) \times \Sigma)$$

and

$$\text{supp } \xi_i^\mathcal{O} \subseteq \Psi_{A,t}(tK_A \sqcup t\Phi_A((R, t^\nu) \times \Sigma)),$$

again using the identifications as before. We can achieve this essentially by cutting off the form $\beta_i^\mathcal{O}$ using $f^\mathcal{O}$. Notice that $\xi_i^\mathcal{O}$ is closed except on the interpolation region between tA and N in $\tilde{N}(t)$.

Definition 4.6 Let $\tilde{\mathcal{K}}^\mathcal{I}(t) = \{\xi_1^\mathcal{I}, \dots, \xi_{m_\mathcal{I}}^\mathcal{I}\}$ and let $\tilde{\mathcal{K}}^\mathcal{O}(t) = \{\xi_1^\mathcal{O}, \dots, \xi_{m_\mathcal{O}}^\mathcal{O}\}$.

Combining Definitions 4.3, 4.4 and 4.6 leads to our approximate kernel.

Definition 4.7 Let $\tilde{\mathcal{K}}_{\text{ap}}(t) = \tilde{\mathcal{K}}_{\text{ap}}^A(t) \oplus \tilde{\mathcal{K}}_{\text{ap}}^N(t) \oplus \tilde{\mathcal{K}}^\mathcal{I}(t)$.

Observe that, by construction, the sums in $\tilde{\mathcal{K}}_{\text{ap}}(t)$ are direct and therefore that

$$\dim \tilde{\mathcal{K}}_{\text{ap}}(t) = b_+^2(A) + b_+^2(\hat{N}) + \dim(\text{Im } j_2^A \cap \text{Im } j_2^N) = b_+^2(\tilde{N}(t)),$$

using (25) and Theorem 3.12.

We may now state our Sobolev embedding inequality.

Theorem 4.8 Recall ν given in (9). There is a constant $C(\tilde{N})$, independent of t , such that if $\alpha \in L_4^2(\Lambda_+^2 T^* \tilde{N}(t))$ satisfies $\langle \alpha, \xi \rangle_{L^2} = 0$ for all $\xi \in \tilde{\mathcal{K}}_{\text{ap}}(t)$ then

$$\|\alpha\|_{L_{4,-2+\delta,t}^2} \leq C(\tilde{N})t^{-\delta(1-\nu)}\|\text{d}\alpha\|_{L_{3,-3+\delta,t}^2}.$$

Note If we could choose $\delta = 0$, i.e. if $-2 \notin \mathcal{D}$ which is equivalent to $b^1(\Sigma) = 0$, then we have a uniform Sobolev embedding constant independent of t . This tallies with the work in [17].

Proof: The idea of the proof is first to show that the only way that the Sobolev embedding constant can blow up as $t \rightarrow 0$ is if we have a sequence converging to a closed self-dual 2-form on A or N . The construction of $\tilde{\mathcal{K}}_{\text{ap}}(t)$ means that the only non-trivial limit that can occur is an element of $\mathcal{K}^\mathcal{O}$. We can therefore just study the behaviour of forms in $\tilde{\mathcal{K}}^\mathcal{O}(t)$ as $t \rightarrow 0$ to deduce our estimate.

Suppose, for a contradiction, that there exists a decreasing sequence of positive numbers $t_n \rightarrow 0$ and a sequence

$$\alpha_n \in L_{4,-2+\delta,t_n}^2(\Lambda_+^2 T^* \tilde{N}(t_n))$$

such that $\langle \alpha_n, \xi \rangle_{L^2} = 0$ for all $\xi \in \tilde{\mathcal{K}}_{\text{ap}}(t_n)$ and

$$\|\alpha_n\|_{L_{4,-2+\delta,t_n}^2} = 1 \geq n t_n^{-\delta(1-\nu)} \|\mathrm{d}\alpha_n\|_{L_{3,-3+\delta,t_n}^2}. \quad (27)$$

Therefore, the sequences

$$\Psi_{A,t_n}^* \alpha_n \in L_{4,-2+\delta}^2(\Lambda_+^2 T^* A) \quad \text{and} \quad \Psi_{N,t_n}^* \alpha_n \in L_{4,-2+\delta}^2(\Lambda_+^2 T^* \hat{N})$$

are bounded. So, by the compact embedding theorem for weighted Sobolev spaces [14, Theorem 4.9], after passing to a subsequence, both sequences converge in $L_{3,-2+\delta'}^2$, to ξ^A and ξ^N say, where $\delta' > \delta$ for ξ^A and $\delta' < \delta$ for ξ^N .

Using the bounds for $\|\mathrm{d}\alpha_n\|_{L_{3,-3+\delta,t_n}^2}$ we see that

$$\|\mathrm{d}(\Psi_{A,t_n}^* \alpha_n)\|_{L_{2,-3+\delta'}^2} \leq \|\mathrm{d}(\Psi_{A,t_n}^* \alpha_n)\|_{L_{3,-3+\delta}^2} \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Hence $\mathrm{d}\xi^A = 0$ and similarly $\mathrm{d}\xi^N = 0$. Elliptic regularity implies that ξ^A and ξ^N are smooth and, as $-2 + \delta \notin \mathcal{D}$, the work in [15] shows that the space of closed self-dual 2-forms is the same at rates $-2 + \delta$ and $-2 + \delta'$ if δ' is sufficiently close to δ . We conclude that $\Psi_{A,t_n}^* \alpha_n$ and $\Psi_{N,t_n}^* \alpha_n$ converge in $L_{4,-2+\delta}^2$ to ξ^A and ξ^N respectively.

The fact that α_n is L^2 -orthogonal to $\tilde{\mathcal{K}}_{\text{ap}}(t_n)$ means that ξ^A is L^2 -orthogonal to $\mathcal{K}_{\text{ap}}^A \oplus \mathcal{K}^{\mathcal{I}}$ and ξ^N is L^2 -orthogonal to $\mathcal{K}_{\text{ap}}^N$. From Proposition 4.2 we deduce that $\xi^N = 0$. If ξ^A is L^2 -orthogonal to $\mathcal{K}^{\mathcal{O}}$ then $\xi^A = 0$ by Proposition 4.1, so we must have that $\xi^A \in \mathcal{K}^{\mathcal{O}}$.

Overall α_n is a sequence of forms such that $\Psi_{N,t_n}^* \alpha_n \rightarrow 0$ and $\Psi_{A,t_n}^* \alpha_n \rightarrow \xi^A \in \mathcal{K}^{\mathcal{O}}$. Thus, for all n sufficiently large, α_n is well approximated by elements in $\tilde{\mathcal{K}}^{\mathcal{O}}(t_n)$, so we now analyse these forms.

By definition, elements of $\mathcal{K}^{\mathcal{O}}$ are kernel forms on A which do *not* extend to corresponding kernel forms on N and thus do not define kernel forms on $\tilde{N}(t)$. Hence, $\tilde{\mathcal{K}}^{\mathcal{O}}(t)$ must be transverse to the closed self-dual 2-forms on $\tilde{N}(t)$ for τ small. Therefore there exist some (t -dependent) constants $C_t(\tilde{N}) > 0$ such that, for all $\alpha \in \tilde{\mathcal{K}}^{\mathcal{O}}(t)$,

$$\|\alpha\|_{L_{4,-2+\delta,t}^2} \leq C_t(\tilde{N}) \|\mathrm{d}\alpha\|_{L_{3,-3+\delta,t}^2}.$$

Recall Definition 4.6 and the discussion preceding it. Any $\alpha \in \tilde{\mathcal{K}}^{\mathcal{O}}(t)$ is identified with $f^{\mathcal{O}}\beta$ for some $\beta \in \mathcal{K}^{\mathcal{O}}$. Using the definition of $f^{\mathcal{O}}$, the facts that

$d\beta = 0$ and satisfies $|\nabla_C^j \Phi_A^* \beta| = O(r^{-2-j})$ for all $j \in \mathbb{N}$ as $r \rightarrow \infty$, together with the assumption that $\delta < 1$ in (20) we calculate

$$\begin{aligned} \|d(f^\mathcal{O} \beta)\|_{L_{3,-3+\delta}^2}^2 &= \|df^\mathcal{O} \wedge \beta\|_{L_{3,-3+\delta}^2}^2 \\ &= \sum_{j=0}^3 \int_{\Phi_A((\frac{1}{2}t^{\nu-1}, t^{\nu-1}) \times \Sigma)} |r^{j+3-\delta} \nabla^j (df^\mathcal{O} \wedge \beta)|^2 r^{-4} d\text{vol}_{g_0|_A} \\ &= O\left(\int_{\frac{1}{2}t^{\nu-1}}^{t^{\nu-1}} |r^{3-\delta} t^{1-\nu} r^{-2}|^2 r^{-1} dr\right) \\ &= O(t^{2\delta(1-\nu)}). \end{aligned}$$

Notice that this norm tends to zero as $t \rightarrow 0$ as we would expect. We deduce that

$$\|\alpha\|_{L_{4,-2+\delta,t}^2} \leq Ct^{-\delta(1-\nu)} \|d\alpha\|_{L_{3,-3+\delta,t}^2}$$

for some constant $C > 0$.

Hence, for $n \gg 1$, there exists some other constant $C > 0$ so that

$$\|\alpha_n\|_{L_{4,-2+\delta,t_n}^2} = 1 \leq Ct_n^{-\delta(1-\nu)} \|d\alpha_n\|_{L_{3,-3+\delta,t_n}^2}.$$

This contradicts (27). \square

Using our estimate we have the following crucial result.

Theorem 4.9 *Recall Definition 4.7. The exterior derivative*

$$\begin{aligned} d : \tilde{\mathcal{K}}_{\text{ap}}(t)^\perp &\subseteq L_{4,-2+\delta,t}^2(\Lambda_+^2 T^* \tilde{N}(t)) \\ &\longrightarrow \{\xi \in L_{3,-3+\delta,t}^2(\Lambda^3 T^* \tilde{N}(t)) : \xi \text{ is exact}\} \end{aligned}$$

is a bounded invertible linear map between Banach spaces with bounded linear inverse P_d satisfying $\|P_d\| \leq C(\tilde{N})t^{-\delta(1-\nu)}$.

Proof: Recall that $\text{Im } d|_{\Lambda_+^2} = \text{Im } d|_{\Lambda^2}$ on compact Riemannian 4-manifolds (c.f. [17, Proposition 2.10]). An immediate consequence of Theorem 4.8 is that the L^2 -orthogonal projection of $\mathcal{H}_+^2(\tilde{N}(t))$, given in Definition 2.11, to $\tilde{\mathcal{K}}_{\text{ap}}(t)$ is injective. It is also surjective since the dimensions of the two spaces are equal by Theorem 3.12.

The domain and range of d given are clearly Banach spaces and the existence of the bounded inverse P_d is now clear. The bound on the operator norm of P_d is simply a restatement of the estimate in Theorem 4.8. \square

4.2 The contraction map

Recall the tubular neighbourhood constructions for C , A and \hat{N} in Propositions 2.8-2.9 which identified nearby deformations with graphs of self-dual 2-forms. Using the isomorphism $\nu(\tilde{N}(t)) \cong \Lambda_+^2 T^* \tilde{N}(t)$ given by Proposition 3.9, a straightforward adaptation of the work in [17, §5.1-5.2] shows that we can construct a tubular neighbourhood $\tilde{T}(t)$ of $\tilde{N}(t)$ in M which is identified with the $\tilde{\epsilon}$ -ball about the zero section in $C_{1,t}^1(\Lambda_+^2 T^* \tilde{N}(t))$ for some $\tilde{\epsilon} > 0$, in a manner which is compatible with the constructions in Propositions 2.8-2.9. Rather than repeating the details here we refer the interested reader to [17, §5.1-5.2]. We can use this construction to describe coassociative deformations of $\tilde{N}(t)$.

Definition 4.10 Let $\alpha \in C^1(\Lambda_+^2 T^* \tilde{N}(t))$ with $\|\alpha\|_{C_{1,t}^1} < \tilde{\epsilon}$. Using the construction discussed above, we can define a nearby deformation $\tilde{N}_\alpha(t) \subseteq \tilde{T}(t)$ of $\tilde{N}(t)$ with a natural diffeomorphism $f_\alpha(t) : \tilde{N}(t) \rightarrow \tilde{N}_\alpha(t)$. Let

$$F_t(\alpha) = f_\alpha(t)^* \left(\varphi|_{\tilde{N}_\alpha(t)} \right).$$

By the coassociativity of N and A , $F_t(\alpha)$ is exact.

By construction, the zeros of F_t correspond exactly to nearby coassociative deformations of $\tilde{N}(t)$.

As in [17, Proposition 6.2], we can say more about the deformation map F_t .

Proposition 4.11 For $\alpha \in C^1(\Lambda_+^2 T^* \tilde{N}(t))$ with $\|\alpha\|_{C_{1,t}^1} < \tilde{\epsilon}$, we may write

$$F_t(\alpha) = \varphi|_{\tilde{N}(t)} + d\alpha + Q_t(\alpha)$$

for a smooth map Q_t depending on α and $\nabla\alpha$. Moreover, if $\alpha \in L_4^2(\Lambda_+^2 T^* \tilde{N}(t))$ with $\|\alpha\|_{C_{1,t}^1} < \tilde{\epsilon}$, then

$$Q_t(\alpha) \in d(L_4^2(\Lambda_+^2 T^* \tilde{N}(t))) \subseteq L_3^2(\Lambda^3 T^* \tilde{N}(t)).$$

From Proposition 4.11 we see that solving $F_t(\alpha) = 0$ is equivalent to solving

$$d\alpha = -\varphi|_{\tilde{N}(t)} - Q_t(\alpha).$$

Since the right-hand side lies in $d(L_4^2(\Lambda_+^2 T^* \tilde{N}(t)))$, we can use Theorem 4.9 and try to solve

$$\alpha = P_d(-\varphi|_{\tilde{N}(t)} - Q_t(\alpha)) \tag{28}$$

for $\alpha \in \tilde{\mathcal{K}}_{\text{ap}}(t)^\perp \subseteq L_4^2(\Lambda_+^2 T^* \tilde{N}(t))$. The idea is to apply the Contraction Mapping Theorem to give a solution to (28), which will in turn define a zero of F_t and hence a coassociative deformation of $\tilde{N}(t)$.

Definition 4.12 For $\alpha \in \tilde{\mathcal{K}}_{\text{ap}}(t)^\perp \subseteq L_4^2(\Lambda_+^2 T^* \tilde{N}(t))$ with $\|\alpha\|_{C_{1,t}^1} < \tilde{\epsilon}$, define

$$\mathcal{C}_t(\alpha) = P_d(-\varphi|_{\tilde{N}(t)} - Q_t(\alpha)) \in \tilde{\mathcal{K}}_{\text{ap}}(t)^\perp \subseteq L_4^2(\Lambda_+^2 T^* \tilde{N}(t)).$$

As observed, fixed points of \mathcal{C}_t define elements of $\text{Ker } F_t$. Moreover, given a fixed point α of \mathcal{C}_t , we may apply the Implicit Function Theorem and parameterise the elements of $\text{Ker } F_t$ near α by $\tilde{\mathcal{K}}_{\text{ap}}(t) \cong \mathcal{H}_+^2(\tilde{N}(t))$.

Given the estimate on the norm of P_d in Theorem 4.9, to show that \mathcal{C}_t is a contraction on some neighbourhood of zero in $L_{4,-2+\delta,t}^2$, it is enough to obtain estimates on the $L_{3,-3+\delta,t}^2$ norm of $\varphi|_{\tilde{N}(t)}$ and $Q_t(\alpha) - Q_t(\beta)$ for $\alpha, \beta \in L_4^2$.

We begin with the estimate on the norm of $\varphi|_{\tilde{N}(t)}$.

Proposition 4.13 *There exists a constant $C(\varphi) > 0$, independent of t , such that*

$$\|\varphi|_{\tilde{N}(t)}\|_{L_{3,-3+\delta,t}^2} \leq C(\varphi) t^{\nu(3-\delta)+2(1-\lambda)(1-\nu)}.$$

Remark It is in the proof of this proposition that we finally use the constraint on ν in (9).

The key idea in the proof is that our matching condition and the assumption that $\lambda < -\frac{1}{2}$ ensure that the terms which should naively give the largest contribution to $|\varphi|_{\tilde{N}(t)}|$ in fact are zero or effectively cancel.

Proof: For convenience, we use the diffeomorphisms given in (23), (24) and (26) to decompose $\tilde{N}(t)$ into three pieces:

$$\begin{aligned} \tilde{N}_l(t) &= \Psi_{A,t}(tK_A \cup t\Phi_A((R, \tfrac{1}{2}t^{\nu-1}) \times \Sigma)) \subseteq \chi(t\hat{A}(t)), \\ \tilde{N}_m(t) &= \Psi_{C,t}([\tfrac{1}{2}t^\nu, t^\nu] \times \Sigma) \text{ and } \tilde{N}_u(t) = \Psi_{N,t}(\Psi_N((t^\nu, \epsilon) \times \Sigma) \cup K_N). \end{aligned}$$

First observe trivially that

$$\|\varphi|_{\tilde{N}(t)}\|_{L_{3,-3+\delta,t}^2}^2 = \|\varphi\|_{L_{3,-3+\delta,t}^2}^2(\tilde{N}_l(t)) + \|\varphi\|_{L_{3,-3+\delta,t}^2}^2(\tilde{N}_m(t)) + \|\varphi\|_{L_{3,-3+\delta,t}^2}^2(\tilde{N}_u(t)). \quad (29)$$

We begin with estimating the norm of φ on $\tilde{N}_l(t)$. Since $\chi^*(\varphi)$ agrees with φ_0 at 0, we have that $\chi^*(\varphi) = \varphi_0 + O(r)$ on $B(0; \epsilon_M)$. In fact, $\chi^*(\nabla^j \varphi) = \nabla^j \varphi_0 + O(r^{1-j})$ for $j \in \mathbb{N}$. Since $\varphi_0|_{t\hat{A}(t)} \equiv 0$, we have that

$$|r^{j+3-\delta} \chi^*(\nabla^j \varphi)| = O(r^{4-\delta}) \quad \text{on } t\hat{A}(t).$$

Hence the dominate terms in calculating the norm of φ on $\tilde{N}_l(t)$ arise on $\chi(t\hat{A}(t)) \setminus \chi(tK_A)$. We may calculate

$$\begin{aligned} \sum_{j=0}^3 \int_{t\Phi_A((R, \frac{1}{2}t^{\nu-1}) \times \Sigma)} |r^{j+3-\delta} \chi^*(\nabla^j \varphi)|^2 r^{-4} d\text{vol}_{g_0|_{tA}} &= O\left(\int_{tR}^{\frac{1}{2}t^{\nu}} r^{2(4-\delta)} r^{-1} dr\right) \\ &= O(t^{2\nu(4-\delta)}). \end{aligned}$$

Hence, there exists a t -independent constant $C_l > 0$ such that

$$\|\varphi\|_{L_{3,-3+\delta,t}^2(\tilde{N}_l(t))}^2 \leq C_l t^{2\nu(4-\delta)}. \quad (30)$$

For $p \in \tilde{N}_u(t)$, because $\varphi|_{\tilde{N}} \equiv 0$, we may decompose $\varphi(p)$ in a similar manner to (6):

$$\varphi(p) = d\alpha_N^0(t)(p) + P_N(p, \alpha_N^0(t)(p), \nabla \alpha_N^0(t)(p)).$$

Since $\alpha_N^0(t) = \sum_{i=1}^d t^{1-\lambda_i} \alpha_N^i$ is closed and $P_N(p, \alpha(p), \nabla \alpha(p))$ is dominated by $|r^{-1} \Psi_N^* \alpha(r, \sigma)|^2$ and $|\nabla_C \Psi_N^* \alpha(r, \sigma)|^2$, we see that the largest contribution to $|\varphi|_{\tilde{N}_u(t)}$ arises from $\Psi_{N,t}(\Psi_N((t^\nu, \epsilon) \times \Sigma))$. From our matching condition in Definition 3.4 we have that $|\nabla_C^j \Psi_N^* \alpha_N^i| = O(r^{\lambda_i-j})$, so on $\tilde{N}_u(t)$ we have $|\varphi| = O(\sum_{i=1}^d t^{2(1-\lambda_i)} r^{2\lambda_i-2})$. We calculate

$$\begin{aligned} \int_{\Psi_N((t^\nu, \epsilon) \times \Sigma)} |r^{3-\delta} \sum_{i=1}^d t^{2(1-\lambda_i)} r^{2\lambda_i-2}|^2 r^{-4} d\text{vol}_{g_\varphi|_{\tilde{N}}} \\ = O\left(\sum_{i=1}^d t^{4(1-\lambda_i)} \int_{t^\nu}^\epsilon r^{4\lambda_i+1-2\delta} dr\right) \\ = O\left(\sum_{i=1}^d t^{\nu(4\lambda_i+2-2\delta)+4(1-\lambda_i)}\right). \end{aligned}$$

Since $\lambda_i \leq \lambda$ for all i , we see that there exists a t -independent constant $C_u > 0$ such that

$$\|\varphi\|_{L_{3,-3+\delta,t}^2(\tilde{N}_u(t))}^2 \leq C_u t^{2\nu(3-\delta)+4(1-\lambda)(1-\nu)}. \quad (31)$$

We are now left with $\tilde{N}_m(t)$, which will be the key contribution to calculate. Since the graph of α_N over C defines \hat{N} near z , we can view $\tilde{N}_m(t)$ as the graph of $\beta = \alpha_C(t) - \alpha_N$ over N via Proposition 2.9. Since N is coassociative and we can approximate the metric on $\tilde{N}_m(t)$ using the conical metric, we see that

$$|\varphi|_{\tilde{N}_m(t)} \leq c|d\beta + P_C(\beta, \nabla_C \beta)|$$

for some $c > 0$ independent of t . Now we can use (10) to decompose β into four terms, which we can estimate using the fact that $2\lambda - 1 < -2$ in Proposition

3.3 and the matching condition as follows:

$$\begin{aligned} |t^3 \nabla_C^j \alpha_A^0(t^{-1}r, \sigma)| &= O\left(\sum_{i=1}^d t^3 \cdot t^{-j-2} (t^{-1}r)^{\lambda_i-j}\right) \\ &= O(t^{1-\lambda_-} r^{\lambda_-j}), \end{aligned} \quad (32)$$

$$|t^3 \nabla_C^j \alpha'_A(t^{-1}r, \sigma)| = O(t^{1-\lambda_-} r^{\lambda_-j}), \quad (33)$$

$$\begin{aligned} |t^{1-\lambda_i} \nabla_C^j (\Psi_N^* \alpha_N^i(r, \sigma) - \alpha_A^i(r, \sigma))| &= O(t^{1-\lambda_i} r^{\lambda_i+\delta_0-j}) \\ &= O(t^{1-\lambda_-} r^{\lambda_0-j}), \end{aligned} \quad (34)$$

$$|\nabla_C^j \alpha_N(r, \sigma)| = O(r^{\mu-j}), \quad (35)$$

where $\lambda_- < -2$. Overall, $|r^{-1}\beta(r, \sigma)|^2$ and $|\nabla_C \beta(r, \sigma)|^2$ are dominated by terms of order $O((t^{-1}r)^{2(\lambda-1)})$ and $O(r^{2(\mu-1)})$. The choice of ν in (9) ensures that $\nu > (1-\lambda)/(\mu-\lambda)$ since $\mu > 1 + \delta_0$ by assumption (8), hence $(1-\lambda)(1-\nu) < \nu(\mu-1)$ so that terms with the former exponent give the greatest contribution as $t \rightarrow 0$ on $\tilde{N}_m(t)$ (recalling that $r = O(t^\nu)$). Thus, we see from (10) that

$$|P_C(r, \sigma, \beta(r, \sigma), \nabla_C \beta(r, \sigma))| = O((t^{-1}r)^{2(\lambda-1)}). \quad (36)$$

Recall that α_A^0 , α_N^i and α_A^i are all closed forms, so we have from (10) that

$$\begin{aligned} d\beta(r, \sigma) &= t^3(1 - f_{\text{inc}}(2t^{-\nu}r - 1))d\alpha'_A(t^{-1}r, \sigma) - 2t^{3-\nu}df_{\text{inc}}(2t^{-\nu}r - 1) \wedge \alpha'_A(t^{-1}r, \sigma) \\ &\quad + 2t^{-\nu}df_{\text{inc}}(2t^{-\nu}r - 1) \wedge \sum_{i=1}^d t^{1-\lambda_i}(\Psi_N^* \alpha_N^i(r, \sigma) - \alpha_A^i(r, \sigma)) \\ &\quad + (f_{\text{inc}}(2t^{-\nu}r - 1) - 1)d\alpha_N(r, \sigma) + 2t^{-\nu}df_{\text{inc}}(2t^{-\nu}r - 1) \wedge \alpha_N(r, \sigma). \end{aligned}$$

From (32)-(35), we may calculate

$$|d\beta(r, \sigma)| \leq c'(t^{1-\lambda_-} r^{\lambda_- - 1} + t^{1-\nu-\lambda_-} r^{\lambda_-} + t^{1-\nu-\lambda_-} r^{\lambda_0+\delta_0} + r^{\mu-1} + t^{-\nu} r^\mu)$$

for some $c' > 0$ independent of r , σ and t . The dominant terms are therefore of order $O(t^{1-\nu-\lambda_-} r^{\lambda_0+\delta_0})$ and $O(r^{\mu-1})$. Again using (9) and recalling that $r = O(t^\nu)$ we find that the former terms dominate since $\mu - \delta_0 > 1 + \delta_0$ by (8) so $1 - \lambda + \nu(\lambda + \delta_0 - 1) < \nu(\mu - 1)$. Thus we have that

$$|d\beta(r, \sigma)| = O(t^{1-\nu-\lambda_-} r^{\lambda_0+\delta_0}). \quad (37)$$

Finally, we can compare (36) and (37). We see by (9) that $2(1-\lambda)(1-\nu) < 1 - \lambda + \nu(\lambda + \delta_0 - 1)$. Thus the terms in (36) are dominant as $t \rightarrow 0$, which is crucial for our later argument. We can therefore estimate

$$\int_{\frac{1}{2}t^\nu}^{t^\nu} |r^{3-\delta} (t^{-1}r)^{2(\lambda-1)}|^2 r^{-1} dr = O(t^{2\nu(3-\delta)+4(1-\lambda)(1-\nu)}).$$

Thus there exists a t -independent constant $C_m > 0$ such that

$$\|\varphi\|_{L_{3,-3+\delta,t}^2(\tilde{N}_m(t))}^2 \leq C_m t^{2\nu(3-\delta)+4(1-\lambda)(1-\nu)}. \quad (38)$$

Combining (29), (30), (31) and (38) gives the result. \square

Following [17, Proposition 6.3] we can estimate the norm of Q_t .

Proposition 4.14 *There exists a constant $C(Q)$, independent of t , such that if $\alpha, \beta \in L_4^2(\Lambda_+^2 T^* \tilde{N}(t))$ with $\|\alpha\|_{C_{1,t}^1}, \|\beta\|_{C_{1,t}^1} < \tilde{\epsilon}$ then*

$$\begin{aligned} \|Q_t(\alpha) - Q_t(\beta)\|_{L_{3,-3+\delta,t}^2} \\ \leq C(Q) t^{-3+\delta} \|\alpha - \beta\|_{L_{4,-2+\delta,t}^2} (\|\alpha\|_{L_{4,-2+\delta,t}^2} + \|\beta\|_{L_{4,-2+\delta,t}^2}). \end{aligned} \quad (39)$$

Before proving this we have the following lemma, which explains the appearance of the factor of $t^{-3+\delta}$ in Proposition 4.14 as the t -dependence of the Sobolev embedding constant between weighted Sobolev spaces of different weights.

Lemma 4.15 *Let $\alpha \in L_4^2(\Lambda_+^2 T^* \tilde{N}(t))$. Then $\alpha \in C^1(\Lambda_+^2 T^* \tilde{N}(t))$ and there exists a constant $c > 0$, independent of α and t , such that*

$$\|\alpha\|_{C_{1,t}^1} \leq c t^{-3+\delta} \|\alpha\|_{L_{4,-2+\delta,t}^2}.$$

Proof: The Sobolev Embedding Theorem gives a continuous embedding $L_4^2 \hookrightarrow C^1$. Examination of the definition of the weighted norms shows there exists a t -independent constant c_0 such that, for all $\alpha \in L_4^2$,

$$\|\alpha\|_{C_{-2+\delta,t}^1} \leq c_0 \|\alpha\|_{L_{4,-2+\delta,t}^2}; \quad (40)$$

i.e. the embedding constant $L_{4,-2+\delta,t}^2 \hookrightarrow C_{-2+\delta,t}^1$ is independent of t .

We now calculate

$$\begin{aligned} \|\alpha\|_{C_{1,t}^1} &= \sup(|\rho_t^{-1}\alpha| + |\nabla\alpha|) = \sup(|\rho_t^{-3+\delta}\rho_t^{2-\delta}\alpha| + |\rho_t^{-3+\delta}\rho_t^{3-\delta}\nabla\alpha|) \\ &\leq c_1 t^{-3+\delta} \|\alpha\|_{C_{-2+\delta,t}^1} \end{aligned} \quad (41)$$

for some constant c_1 independent of t and α . Combining (40) and (41) proves the lemma. \square

Proof of Proposition 4.14. In the proof of [17, Proposition 6.2] and following [17, Proposition 6.3], it is explained that

$$|Q_t(\alpha) - Q_t(\beta)| = O\left((|\rho_t^{-1}(\alpha - \beta)| + |\nabla(\alpha - \beta)|)(|\rho_t^{-1}\alpha| + |\nabla\alpha| + |\rho_t^{-1}\beta| + |\nabla\beta|)\right); \quad (42)$$

that is, Q_t is dominated by quadratic terms in $\rho_t^{-1}\alpha$ and $\nabla\alpha$ when $\|\alpha\|_{C_{1,t}^1} < \tilde{\epsilon}$. Therefore, an inequality of the type (39) must hold for some (possibly t -dependent) constant $C(Q)$ (see, for example, [9, Proposition 5.8] for a detailed description of the type of argument involved). It suffices therefore to show that $C(Q)$ can be chosen to be independent of t .

We may calculate using (42) with $\beta = 0$ to show that

$$\begin{aligned} \|Q_t(\alpha)\|_{L_{0,-3+\delta,t}^2}^2 &= \int_{\tilde{N}(t)} \rho_t^{2(3-\delta)} |Q_t(\alpha)|^2 \rho_t^{-4} \mathrm{dvol}_{\tilde{g}(t)} \\ &= O\left(\|\alpha\|_{C_{1,t}^1}^2 \int_{\tilde{N}(t)} \rho_t^{2(2-\delta)} \rho_t^2 (|\rho_t^{-1}\alpha| + |\nabla\alpha|)^2 \rho_t^{-4} \mathrm{dvol}_{\tilde{g}(t)}\right) \\ &= O(\|\alpha\|_{C_{1,t}^1}^2 \|\alpha\|_{L_{1,-2+\delta,t}^2}^2). \end{aligned} \quad (43)$$

Using Lemma 4.15 and (43) shows that

$$\|Q_t(\alpha)\|_{L_{0,-3+\delta,t}^2} \leq c_2 t^{-3+\delta} \|\alpha\|_{L_{4,-2+\delta,t}^2}^2 \quad (44)$$

for some t -independent constant c_2 . We deduce that (44) can be improved to give (39) with constant $C(Q)$ independent of t . \square

We may now show that \mathcal{C}_t is indeed a contraction.

Theorem 4.16 *Let*

$$3\nu - \delta + 2(1 - \lambda)(1 - \nu) > \kappa > 3\nu - \delta + (3 + \delta)(1 - \nu) = 3 - \nu\delta$$

(which is possible since $\nu < 1$ and $2(1 - \lambda) > 3 + \delta$ by (20)) and let

$$B_{t^\kappa} = \{\alpha \in \tilde{\mathcal{K}}_{\mathrm{ap}}(t)^\perp \subseteq L_4^2(\Lambda_+^2 T^* \tilde{N}(t)) : \|\alpha\|_{L_{4,-2+\delta,t}^2} \leq t^\kappa\}.$$

Then $\mathcal{C}_t : B_{t^\kappa} \rightarrow B_{t^\kappa}$ has a unique fixed point $\tilde{\alpha}(t)$.

Proof: Let $\alpha, \beta \in B_{t^\kappa}$. By Lemma 4.15

$$\|\alpha\|_{C_{1,t}^1} \leq c t^{-3+\delta+\kappa},$$

so we first choose τ such that $c\tau^{-3+\delta+\kappa} < \tilde{\epsilon}$ so that $F_t(\alpha)$, and thus $\mathcal{C}_t(\alpha)$, is well-defined. This is possible by choice of $\kappa > 3 - \nu\delta > 3 - \delta$.

Using Theorem 4.9, Proposition 4.13 and Proposition 4.14, we calculate

$$\begin{aligned} \|\mathcal{C}_t(\alpha)\|_{L_{4,-2+\delta,t}^2} &= \|P_d(-\varphi|_{\tilde{N}(t)} - Q_t(\alpha))\|_{L_{4,-2+\delta,t}^2} \\ &\leq C(\tilde{N}) t^{-\delta(1-\nu)} (\|\varphi|_{\tilde{N}(t)}\|_{L_{3,-3+\delta,t}^2} + \|Q_t(\alpha)\|_{L_{3,-3+\delta,t}^2}) \\ &\leq C(\tilde{N}) t^{-\delta(1-\nu)} (C(\varphi) t^{\nu(3-\delta)+2(1-\lambda)(1-\nu)} + C(Q) t^{-3+\delta} \|\alpha\|_{L_{4,-2+\delta,t}^2}^2). \end{aligned}$$

Taking τ such that

$$C(\tilde{N})(C(\varphi)\tau^{3\nu-\delta+2(1-\lambda)(1-\nu)-\kappa} + C(Q)\tau^{-3+\nu\delta+\kappa}) < 1,$$

which is possible by the choice of κ , ensures that $\mathcal{C}_t(\alpha) \in B_{t^\kappa}$.

Using Theorem 4.9 and Proposition 4.14 again, we deduce that

$$\begin{aligned} & \|\mathcal{C}_t(\alpha) - \mathcal{C}_t(\beta)\|_{L_{4,-2+\delta,t}^2} \\ &= \|P_d(Q_t(\alpha) - Q_t(\beta))\|_{L_{4,-2+\delta,t}^2} \\ &\leq C(\tilde{N})C(Q)t^{-3+\nu\delta}\|\alpha - \beta\|_{L_{4,-2+\delta,t}^2}(\|\alpha\|_{L_{4,-2+\delta,t}^2} + \|\beta\|_{L_{4,-2+\delta,t}^2}). \end{aligned}$$

We finally take τ such that

$$2C(\tilde{N})C(Q)\tau^{-3+\nu\delta+\kappa} < 1$$

so that $\mathcal{C}_t : B_{t^\kappa} \rightarrow B_{t^\kappa}$ is a contraction. Applying the Contraction Mapping Theorem gives the result. \square

Our main result (Theorem 1.2) now follows from the next theorem.

Theorem 4.17 *For all $t \in (0, \tau)$, let $N(t) = \tilde{N}_{\tilde{\alpha}(t)}(t)$ as in Definition 4.10 with $\tilde{\alpha}(t)$ given by Theorem 4.16. Then $N(t)$ is a smooth compact coassociative 4-fold such that $N(t) \rightarrow N$ as $t \rightarrow 0$ in the sense of currents.*

Proof: Since $L_4^2 \hookrightarrow C^{1,a}$ by the Sobolev Embedding Theorem, we can apply the method of proof of [17, Proposition 7.16] to show that $\tilde{\alpha}(t)$ is smooth. The result is now immediate by definition of $\tilde{N}(t)$. \square

We can now deduce Corollary 1.3 since we can parameterize the zeros of F_t near $\tilde{\alpha}(t)$ using closed self-dual 2-forms on $\tilde{N}(t)$.

Proposition 4.18 *There is a smooth family of compact coassociative smoothings of N of dimension $b_+^2(A) + b_+^2(\hat{N}) + \dim(\text{Im } j_2^A \cap \text{Im } j_2^N)$.*

5 Applications

In this section we give some applications of our main results. We first use these results to describe the relationship between the moduli space of “matching pairs” of AC and CS coassociative 4-folds which can be used in our desingularization and the moduli space of the smooth compact coassociative 4-fold we construct. This work leads us to deduce Proposition 1.4 which gives evidence, in the stable case, for local surjectivity of our gluing; i.e. that all nearby smooth coassociative 4-folds to the given CS coassociative 4-fold arise from our desingularization method. We then discuss examples where our theory applies and consequences.

5.1 Moduli spaces

We now describe how the moduli spaces of the CS and AC building blocks “fit together” with the moduli space of smoothings, in a similar manner to [8, §8].

Suppose we have an almost G_2 manifold M and a matching pair of a CS coassociative 4-fold $N \subseteq M$ and an AC coassociative 4-fold $A \subseteq \mathbb{R}^7$ with asymptotic cone $C \cong \mathbb{R}^+ \times \Sigma$ to which Theorem 1.2 applies. Hence we have $\tau > 0$ and smooth compact coassociative 4-folds $N(t)$ for $t \in (0, \tau)$ such that $N(t) \rightarrow N$ as $t \rightarrow 0$. We can make τ canonical by taking the supremum, which will be finite. Moreover, since $t \in (0, \tau)$ determines the scale of A glued into N to form $N(t)$, and we are free to re-scale A initially and maintain the AC convergence to C , we may choose A such that $\tau = 2$. All of the $N(t)$ are diffeomorphic to the same compact coassociative 4-fold, so we set $X = N(1)$ for definiteness.

We make a definition for convenience.

Definition 5.1 Let $\mathcal{M}(N)$ denote the moduli space of CS coassociative deformations of N with cone C and rate μ_0 , where $\mu_0 \leq \mu$ with $(1, \mu_0] \cap \mathcal{D} = \emptyset$.

Recall that for any coassociative 4-fold A' in \mathbb{R}^7 , φ_0 defines an element $[\varphi_0] \in H^2(A') \cong H^3(\mathbb{R}^7; A')$. Let $\mathcal{M}(A)$ denote the moduli space of AC coassociative deformations A' of A with cone C and rate λ_0 , where $\lambda_0 \geq \lambda$ with $[\lambda_0, -\frac{1}{2}] \cap \mathcal{D} = \emptyset$, such that $j_2^{A'}[\varphi_0] \in \text{Im } j_2^A \cap \text{Im } j_2^N \subseteq H^2(\Sigma)$.

Let $\mathcal{M}(X)$ denote the moduli space of compact coassociative deformations of X , the coassociative 4-fold arising from gluing N and A as described above. Theorems 2.12 and 3.12 state that $\mathcal{M}(X)$ is a smooth manifold of dimension $b_+^2(X) = b_+^2(\hat{N}) + b_+^2(A) + \dim(\text{Im } j_2^A \cap \text{Im } j_2^N)$.

The main point of this definition is that pairs $(N', A') \in \mathcal{M}(N) \times \mathcal{M}(A)$ satisfy the topological matching condition as well as the constraint on the AC rate of convergence to C . Therefore, the set of gluing data near (N, A) for which we can apply Theorem 1.2 is a subset of $\mathcal{M}(N) \times \mathcal{M}(A)$. Moreover, we have a natural map from the gluing data into $\mathcal{M}(X)$ given by Theorem 1.2. Since we desingularize N using A to get X it is natural to ask whether we can construct all compact coassociative 4-folds near X via gluing; that is, whether the gluing map is a local diffeomorphism. In general this should not be possible, and the first thing to compare is the dimensions of $\mathcal{M}(N)$, $\mathcal{M}(A)$ and $\mathcal{M}(X)$.

We begin by recalling the description of $\mathcal{M}(N)$ from [16].

Theorem 5.2 *There exist finite-dimensional vector spaces of forms $\mathcal{I}(N)$ and $\mathcal{O}(N)$, an open neighbourhood $\hat{\mathcal{M}}(N)$ of 0 in $\mathcal{I}(N)$ and a smooth map $\pi : \hat{\mathcal{M}}(N) \rightarrow \mathcal{O}(N)$ such that $\mathcal{M}(N)$ near N is locally homeomorphic to $\pi^{-1}(0)$*

near 0. Moreover, the expected dimension of $\mathcal{M}(N)$ is

$$b_+^2(\hat{N}) - \sum_{v \in (-2, -1]} d_{\mathcal{D}}(v) - \text{ind}(C),$$

where the stability index $\text{ind}(C)$ is given in Definition 3.19.

Remarks

- (a) In [16] a lower bound for the expected dimension is given, but Proposition 3.18 allows us to improve the lower bound to an equality here.
- (b) As discussed in [20], it is possible to generalize the deformation theory of N so that the cone at the singularity deforms in a family \mathcal{C} and the analogous result to Theorem 5.2 holds with the stability index replaced by the \mathcal{C} -stability index.

We can easily calculate the expected difference in $\dim \mathcal{M}(X)$ and $\dim \mathcal{M}(N)$:

$$b_+^2(A) + \dim(\text{Im } j_2^A \cap \text{Im } j_2^N) + \sum_{v \in (-2, -1]} d_{\mathcal{D}}(v) + \text{ind}(C).$$

We deduce that the higher the stability index of C , the “less likely” a compact coassociative 4-fold is going to develop a conical singularity modelled on C .

We now use the deformation theory for A from [18] to describe $\mathcal{M}(A)$.

Theorem 5.3 *The space $\mathcal{M}(A)$ is a smooth manifold near A of dimension*

$$b_+^2(A) + \dim(\text{Im } j_2^A \cap \text{Im } j_2^N) + \sum_{v \in (-2, -\frac{1}{2})} d_{\mathcal{D}}(v).$$

Proof: The moduli space $\hat{\mathcal{M}}(A)$ of AC coassociative deformations of A with cone C and rate λ_0 is a smooth manifold by Theorem 3.2. Moreover, the tangent space $T_A \hat{\mathcal{M}}(A)$ is isomorphic to the closed self-dual 2-forms on A in L_{4, λ_0}^2 .

We may define a smooth map $\pi : \hat{\mathcal{M}}(A) \rightarrow \text{Im } j_2^A \subseteq H^2(\Sigma)$ by $\pi(A') = j_2^{A'}[\varphi_0]$. Then $d\pi|_A : T_A \hat{\mathcal{M}}(A) \rightarrow \text{Im } j_2^A$ is surjective by the work in [18], as explained after Theorem 3.2. Thus π is a submersion, so it follows that $\pi^{-1}(\text{Im } j_2^A \cap \text{Im } j_2^N) = \mathcal{M}(A)$ is a smooth manifold of the claimed dimension. \square

Theorems 5.2 and 5.3 show that if $\mathcal{M}(N)$ is smooth of the expected dimension, then

$$\dim \mathcal{M}(N) + \dim \mathcal{M}(A) = \dim \mathcal{M}(X) - \left(\text{ind}(C) - \sum_{v \in (-1, -\frac{1}{2})} d_{\mathcal{D}}(v) \right).$$

We deduce from Definition 3.19 that the quantity in brackets is non-negative and vanishes if and only if C is stable. We conclude that, unless C is stable, our gluing method can only at most generate a subset of $\mathcal{M}(X)$ near X .

We therefore from now on restrict our attention to the situation where C is stable, which corresponds, in some sense, to the most probable type of conical singularity to occur. It follows from Theorem 5.2 that $\mathcal{M}(N)$ is smooth.

Proposition 1.1 and Theorem 1.2 also imply that for any $(N', A') \in \mathcal{M}(N) \times \mathcal{M}(A)$ there exists $\tau(N', A') > 0$ and smooth compact coassociative 4-folds $N'(t)$ for $0 < t < \tau(N', A')$ formed by gluing N' and A' which converge to N' as $t \rightarrow 0$. (We can make $\tau(N', A')$ canonical by taking the supremum again.) Observe further that $A' \in \mathcal{M}(A)$ implies that $tA' \in \mathcal{M}(A)$ for all $t > 0$ and we may choose $\tau(N', A')$ such that $t\tau(N', tA') = \tau(N', A')$. We can thus make the following definition.

Definition 5.4 Using the notation above, let

$$\mathcal{M}(N, A) = \{(N', tA') \in \mathcal{M}(N) \times \mathcal{M}(A) : t \in (0, \tau(N', A'))\},$$

which is the moduli space of “matching pairs”. We can define a smooth map $G : \mathcal{M}(N, A) \rightarrow \mathcal{M}(X)$ by $G(N', tA') = N'(t)$.

Having defined our “gluing map” G , we can show Proposition 1.4.

Proposition 5.5 *The map G is a local diffeomorphism.*

Proof: Consider $dG|_{(N,A)} : T_N\mathcal{M}(N) \oplus T_A\mathcal{M}(A) \rightarrow T_X\mathcal{M}(X)$. Recall that

$$T_X\mathcal{M}(X) \cong \{\alpha \in L_4^2(\Lambda_+^2 T^*X) : d\alpha = 0\}$$

and that the proof of Theorem 2.12 implies that we can use these closed self-dual 2-forms to define natural coordinates on the moduli space $\mathcal{M}(X)$.

A consequence of the work in [16], Theorem 5.3 and the stability of C is that, for $\delta > 0$ such that $(-1, -1 + \delta) \cap \mathcal{D} = \emptyset$, we have

$$\begin{aligned} T_N\mathcal{M}(N) &\cong \{\alpha \in L_{4,-1+\delta}^2(\Lambda_+^2 T^*\hat{N}) : d\alpha = 0\} \quad \text{and} \\ T_A\mathcal{M}(A) &\cong \{\alpha \in L_{4,-1+\delta}^2(\Lambda_+^2 T^*A) : d\alpha = 0, j_2^A[\alpha] \in \text{Im } j_2^N\}. \end{aligned}$$

Moreover, we can use these spaces of closed self-dual 2-forms to define natural coordinates on $\mathcal{M}(N)$ and $\mathcal{M}(A)$ respectively.

In our construction of the approximation of the closed self-dual 2-forms on X in Definition 4.7, we used the same spaces of forms on N and A as above except with the weighted Sobolev space $L_{4,-2+\delta}^2$. The crucial fact was that the

topological condition that $j_2^A[\alpha] \in \text{Im } j_2^N$ enabled us to match closed self-dual 2-forms α on A with decay of order $O(r^{-2})$ to closed self-dual 2-forms on \hat{N} with the same decay, and thus effectively interpolate between them to construct our desired self-dual 2-form which is “almost” closed. Since C is stable, the analytic matching condition in Definition 3.4 is always satisfied, meaning in particular that any closed self-dual form on A of order $O(r^v)$ for $v \in (-2, -1]$ can be matched with a corresponding closed self-dual 2-form on \hat{N} .

Moreover, we have from Theorems 5.2 and 5.3 that

$$\begin{aligned} \dim \mathcal{M}(N) &= b_+^2(\hat{N}) - \sum_{v \in (-2, -1]} d_{\mathcal{D}}(v) \quad \text{and} \\ \dim \mathcal{M}(A) &= b_+^2(A) + \dim(\text{Im } j_2^A \cap \text{Im } j_2^N) + \sum_{v \in (-2, -1]} d_{\mathcal{D}}(v) \end{aligned}$$

since $d_{\mathcal{D}}(v) = 0$ for $v \in (-1, -\frac{1}{2})$ by the stability of C . It therefore follows from Theorem 3.12 that

$$\dim \mathcal{M}(N) + \dim \mathcal{M}(A) = b_+^2(\hat{N}) + b_+^2(A) + \dim(\text{Im } j_2^A \cap \text{Im } j_2^N) = \dim \mathcal{M}(X).$$

We conclude therefore, in the same way as for our approximate kernel in Definition 4.7, that we may define a natural isomorphism between $T_N \mathcal{M}(N) \oplus T_A \mathcal{M}(A)$ and $T_X \mathcal{M}(X)$. We may thus identify the product of the natural coordinates on $\mathcal{M}(N)$ and $\mathcal{M}(A)$ with the natural coordinates on $\mathcal{M}(X)$. With this identification $dG|_{(N,A)}$ becomes the identity map and hence G is a local diffeomorphism. \square

Thus all compact coassociative 4-folds near X arise via gluing in the stable case. Moreover, Proposition 5.5 suggests that all elements of $\mathcal{M}(X)$ “sufficiently close” to $\mathcal{M}(N)$, thought of as lying in the “boundary” of $\mathcal{M}(X)$, arise via the desingularization given by Theorem 1.2.

Remark It is also possible to extend our discussion of the stable case to where C is \mathcal{C} -stable, as long as one knows that for every deformation C' of C in \mathcal{C} there is a corresponding deformation A' of A which is AC to C' .

5.2 Examples

We now wish to discuss applications of our theory in examples. We recall that to apply our results we need

- a coassociative 4-fold N , in an almost G_2 manifold M , with a conical singularity z modelled on a cone $C \cong \mathbb{R}^+ \times \Sigma$ and

- a coassociative 4-fold $A \subseteq \mathbb{R}^7$ asymptotically conical with rate $\lambda < -\frac{1}{2}$ to C

such that A and N satisfy the matching condition given in Definition 3.4.

A particular criterion for when this matching condition is satisfied is given in Proposition 1.1, namely that the topological matching condition holds (see Definition 3.7) and the cone C is \mathcal{C} -stable in the sense of Definition 3.19. Recall that stability is related to the exceptional rates \mathcal{D} given in Definition 3.1. Moreover, in the case when $\lambda \leq -2$ the matching condition is equivalent to the topological matching condition.

We now give examples of situations where we can apply our results and begin with a degenerate case.

Example 5.6 Suppose we make the perverse choice that N is smooth and z is any point. Then $C = \mathbb{R}^4$ and $\Sigma \cong \mathcal{S}^3$ so $b^1(\Sigma) = 0$ and the topological matching condition will hold trivially. We can take $A = \mathbb{R}^4$ which is obviously AC with any negative rate, so the matching condition is satisfied. Since $b_+^2(A) = 0$ and $b_+^2(\hat{N}) = b_+^2(N)$, applying Corollary 1.3 gives that there is a $b_+^2(N)$ -dimensional deformation family of coassociative “smoothings” of N . This corresponds to the fact that our gluing construction will just give back N in this case.

Since \mathbb{R}^4 is stable by [20, Corollary 5.7], Proposition 1.1 shows that Theorem 1.2 would apply to gluing in any asymptotically planar but non-planar A with rate $\lambda < -\frac{1}{2}$ into smooth N . However, we can show that no such A exists.

Proposition 5.7 *If a coassociative 4-fold A in \mathbb{R}^7 is AC with rate $\lambda < 0$ to a coassociative 4-plane C , then $A = C$.*

Proof: Suppose, for a contradiction, that there is a least choice of λ such that $\lambda \geq -2$. Since $\mathcal{D} \cap (-3, 0) = \emptyset$ by [20, Corollary 5.7], it follows from Proposition 3.3 that A can be written as the graph of a self-dual 2-form on C which has decay rate $O(r^{\max\{2\lambda-1, \lambda_-\}})$ where $\lambda_- < -2$. Thus A is also AC with rate $\max\{2\lambda-1, \lambda_-\} < \lambda$, which gives our required contradiction.

Since $\lambda < -2$ and the stabilizer of C in G_2 is $SO(4)$, we can apply [18, Proposition 9.1] and deduce that A is also $SO(4)$ -invariant. However, this $SO(4)$ action decomposes $\mathbb{R}^7 = C \oplus C^\perp = \mathbb{R}^4 \oplus \mathbb{R}^3$, so if A is $SO(4)$ -invariant it must equal C . \square

We now relate our results to the work in [17] and discuss natural extensions.

Example 5.8 In the desingularization theory in [17], we assumed that $b^1(\Sigma) = 0$ and $\lambda < -2$. In this case, the matching condition is equivalent to the topological

matching condition (as $\lambda < -2$), but this is trivially satisfied since $b^1(\Sigma) = 0$. Applying Theorem 1.2 gives nothing but the main result in [17] and we have a deformation family of coassociative smoothings of N of dimension $b_+^2(A) + b_+^2(N)$ by Corollary 1.3.

In fact, if we drop the assumption that $b^1(\Sigma) = 0$, our topological matching condition is still met since $\lambda < -2$ means that α_A^0 , given in Proposition 3.3, vanishes. Hence Theorem 1.2 applies and clearly extends the work in [17].

Example 5.9 If we assume that $b^1(\Sigma) \neq 0$ (so $-2 \in \mathcal{D}$) and $\lambda \leq -2$, our matching condition is potentially non-trivial and equivalent to the topological matching condition. If we then assume further that the topological criterion holds, we may apply Theorem 1.2 and deduce that we can smooth N using A via gluing. This situation is directly analogous to the material in [10] on desingularization of special Lagrangian conical singularities in what is described as “the obstructed case”.

Remark Example 5.9 shows that the work in this article provides an extension of the desingularization theory in the coassociative world which has no current analogue in special Lagrangian geometry, but which should surely follow by adapting the ideas presented here.

Perhaps the best known example of a non-trivial coassociative cone is the Lawson–Osserman $SU(2)$ -invariant cone (see [13] or [20, Example 4.2], for example), originally exhibited because it gives an example of an area-minimizing Lipschitz submanifold which is not smooth. This cone gives a natural model for a coassociative conical singularity.

Example 5.10 Suppose C is the Lawson–Osserman cone. Then $\Sigma \cong \mathcal{S}^3$, $\mathcal{D} \cap (-2, 0) = \{-\frac{3}{2}\}$ and $d_{\mathcal{D}}(-\frac{3}{2}) = 1$ by [20, Corollary 5.8]. Moreover, we have a dilation family of AC smoothings A of C with rate $-\frac{3}{2}$ which have $b_+^2(A) = 0$ (see [4, Theorem IV.3.2] or [18, Proposition 9.3], for example).

Therefore the topological matching condition is trivially satisfied, so the matching condition is equivalent to the dilation deformation of A extending to an infinitesimal deformation of N , which is obviously essential for the smoothings of N to exist via gluing. The stability of C [20, Corollary 5.8] implies that this always occurs by Proposition 1.1.

Applying Corollary 1.3 gives a family of coassociative smoothings X of N of dimension $b_+^2(\hat{N})$. Notice that the stability of C means that N has a smooth moduli space of deformations as a CS coassociative 4-fold of dimension $b_+^2(\hat{N}) - d_{\mathcal{D}}(-\frac{3}{2}) = b_+^2(\hat{N}) - 1$ by Theorem 5.2. Hence, $\dim \mathcal{M}(N) =$

$\dim \mathcal{M}(X) - 1$ (in the notation of Definition 5.1). Moreover, since $\mathcal{M}(A) \cong \mathbb{R}$, the gluing map G given in Definition 5.4 acts between $\mathcal{M}(N) \times (0, \tau)$ and $\mathcal{M}(X)$, and Proposition 1.4 shows that G is a local diffeomorphism. Thus, all nearby compact coassociative deformations of X arise in one-parameter families which degenerate to elements of $\mathcal{M}(N)$.

Remarks In the situation of Example 5.10, it is natural to speculate whether we can view $\mathcal{M}(N)$ as (the top stratum of) the boundary of $\mathcal{M}(X)$; that is, whether we can identify $\mathcal{M}(N) \times [0, \tau)$ with a neighbourhood of N in the compactified moduli space $\overline{\mathcal{M}(X)}$. We can form this compactification by taking the closure in the space of coassociative integral currents (see, for example, [20, §2.1] for a definition), and thus the hope would be to prove that every coassociative integral current close to N is either a CS deformation of N or else arises via gluing. This would be the coassociative analogue of [6].

Using Proposition 3.3 and the classification of $SU(2)$ -invariant coassociative 4-folds in a similar manner to the proof of Proposition 5.7, one may deduce that any coassociative 4-fold AC with rate $\lambda < 0$ to the Lawson–Osserman cone C must have $\lambda = -\frac{3}{2}$. As far as the author is aware, it is an open question whether the $SU(2)$ -invariant coassociative 4-folds given in Example 5.10 are the unique AC coassociative 4-folds asymptotic to C – they are certainly locally unique by the work in [18]. If they are unique, then there is a unique possible method of desingularizing conical singularities modelled on C via gluing, given by Theorem 1.2.

Example 5.11 Fox [3, Example 9.2] generalized the Lawson–Osserman cone to define a cone $C(\Gamma)$ given any (non-totally geodesic) null-torsion pseudoholomorphic curve Γ in \mathcal{S}^6 . (The curves Γ were defined by Bryant [2, §4] – see, for example, [19, §3.2] for a definition. The Lawson–Osserman example corresponds to choosing Γ to be a totally geodesic \mathcal{S}^2 .)

Moreover, $C(\Gamma)$ admits a dilation family of smoothings $A(\Gamma)$ which converge with rate $-\frac{3}{2}$ to (possibly a finite cover of) $C(\Gamma)$ (c.f. [3, Theorem 9.3]). If $C = C(\Gamma)$ and $A = A(\Gamma)$ is AC, then our theory applies whenever the matching condition holds, which is a non-trivial constraint.

Remarks Taking the curve Γ in Example 5.11 to be the constant curvature $\frac{1}{6}$ null-torsion pseudoholomorphic \mathcal{S}^2 in \mathcal{S}^6 (the so-called Borůvka sphere) leads to a cone with constant curvature $\frac{1}{16}$ link $SO(3)/A_4$ (c.f. [19, §6.3]). However, one can check that the end of $A(\Gamma)$ is diffeomorphic to $\mathbb{R}^+ \times SO(3)/\mathbb{Z}_3$ and so $A(\Gamma)$ is *not* AC in our sense, and thus cannot be used in our gluing procedure.

We now conclude with examples arising from complex geometry.

Example 5.12 If C is complex, [20, Theorem 6.5] shows that $(-2, 0) \cap \mathcal{D} = \{-1\}$ and $d_{\mathcal{D}}(-1)$ can be determined by the degree of the holomorphic curve in \mathbb{CP}^2 which is the complex link of C . Thus for any AC smoothing of C either $\lambda \leq -2$ or $\lambda = -1$. We can therefore apply our theory whenever the matching condition holds, which is non-trivial to check in general. A case of particular interest is discussed in the next example.

Remark The same situation as Example 5.12 holds if the link Σ of C is a tube of radius $\frac{\pi}{2}$ in the second normal bundle of a null-torsion pseudoholomorphic curve in \mathcal{S}^6 (see [19, Example 6.12] for a description of such Σ).

Example 5.13 In [20, Theorem 1.3] the author constructed CS coassociative 4-folds N in holonomy G_2 manifolds which are diffeomorphic to $K3$ surfaces, so $b_+^2(\hat{N}) = 3$. The cone C at the singularity is complex with $\Sigma \cong \mathbb{RP}^3$.

There is a 2-parameter family of AC smoothings A of the cone at the singularity which have rate -1 and $b_+^2(A) = 0$. Moreover, $d_{\mathcal{D}}(-1) = 2$, corresponding to the choices for A . Thus, since the topological matching condition is trivially satisfied, the matching condition holds if and only if the 2-parameter family of deformations of A extend to infinitesimal deformations of N .

By [20, Corollary 6.11], C is \mathcal{C} -stable for some natural choice of family of cones \mathcal{C} , so Proposition 1.1 implies that we may apply our theory and Corollary 1.3 gives us a 3-dimensional family of smoothings of N . In fact, since the coassociative 4-folds arise initially in a fibration (see [12] and [20, §7]), we see that this family of smoothings is maximal. Notice also that the \mathcal{C} -stability of C implies that N has a smooth moduli space of CS deformations of dimension $b_+^2(\hat{N}) - d_{\mathcal{D}}(-1) = 1$.

Thus, in the notation of Definition 5.1, $\dim \mathcal{M}(N) = \dim \mathcal{M}(X) - 2$, which agrees with the fact that singular fibres in the fibration arise in \mathcal{S}^1 -families and are thus codimension two in the space of smooth fibres. Moreover, for every deformation of C in \mathcal{C} there is a corresponding deformation of A which is AC to the deformed cone. Therefore, as remarked after Proposition 5.5, the gluing map defines a local diffeomorphism $G : \mathcal{M}(N) \times B(0; \tau) \rightarrow \mathcal{M}(X)$ where $B(0; \tau) \subseteq \mathbb{R}^2$. Thus every smooth fibre near X arises via gluing and comes in a 2-parameter family which degenerates to a singular fibre in $\mathcal{M}(N)$.

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